

On Some Impossible Properties of the Affine Line Object $\underline{\mathbb{A}}_S^1$

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Question

What is a scheme?

Schemes as Locally Ringed Space

Question

What is a scheme?

Answer

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ that is *locally affine*.

Schemes as Locally Ringed Space

Fermat scheme

We give a scheme S by giving

- (i) The set of the prime ideals of $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$,
- (ii) The Zariski topology,
- (iii) The structure sheaf.

Schemes as *Points*

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Question

What is a scheme?

Answer

A scheme S is given by its X -valued *points* for all other schemes X , i.e. by the functor

$$\begin{aligned} S : (\mathrm{Sch})^{\mathrm{Op}} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \mathrm{Hom}_{\mathrm{Sch}}(X, S). \end{aligned}$$

Schemes as *Points*

Question

Why does this work?

Schemes as *Points*

Question

Why does this work?

Answer

By the Yoneda Lemma, we're not losing information.

By the co-Yoneda Lemma, we retain dense generation by affines.

In other words, the functor of points is really a functor

$$S : \mathbf{CRing} \longrightarrow \mathbf{Set}$$

$$A \longmapsto \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(A), S).$$

Schemes as *Points*

Fermat scheme

The Fermat scheme is given by the functor of points

$$A \rightarrow \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\}.$$

Schemes as *Points*

Affine Line

We can realise the forgetful functor $\mathbf{CRing} \rightarrow \mathbf{Set}$ as the functor of points of $\mathbb{Z}[X]$

$$\underline{\mathbb{A}}^1 : \mathbf{CRing} \longrightarrow \mathbf{Set}$$

$$A \longmapsto \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(A), \mathrm{Spec}(\mathbb{Z}[X])).$$

We call $\underline{\mathbb{A}}^1$ the **affine line**.

Schemes as *Points*

Idea

Algebraic geometry is classically the study of zeroes of families of polynomials, a situation generalized by the notion of affine schemes, the formal duals of commutative rings. Although the category $\mathbf{Aff}$ of affine schemes is not closed under many operations, it densely generates the more regular Zariski topos $\mathbf{Sh}(\mathbf{Aff})$ in which the affine schemes become the representables, and the general schemes are the sheaves which are covered by affines via open maps [SA20].

Schemes as Synthetic Sets

Question

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Question

What is a scheme?

Answer

A scheme is a set of points in some non-standard mathematical universe.

Schemes as Synthetic Sets

Fermat scheme

The Fermat scheme should be given by the set

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}.$$

Schemes as Synthetic Sets

Question

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Schemes as Synthetic Sets

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Answer

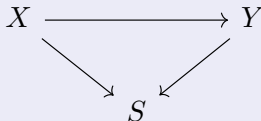
With a heavy dose of category theory!

Defining $\text{Zar}(S)$

We now fix a scheme S and switch to relative scheme theory.

Definition (S -Scheme)

An S -**scheme** X is a scheme X together with a morphism $X \rightarrow S$. A **morphism of S -schemes** X, Y is a morphism of schemes $X \rightarrow Y$ making the triangle commute



Defining $\mathrm{Zar}(S)$

Let Sch/S be the category of S -schemes:

Definition ($\mathrm{Zar}(S)$)

The **big Zariski topos** $\mathrm{Zar}(S)$ is the category of sheaves $\mathrm{Sh}(\mathrm{Sch}/S)$ on the site Sch/S .

Defining $\mathrm{Zar}(S)$

An object of $\mathrm{Zar}(S)$ is a functor $F : (\mathrm{Sch}/S)^{\mathrm{Op}} \rightarrow \mathrm{Set}$ that satisfies the glueing condition with respect to Zariski covers: if $X = \bigcup_{i \in I} U_i$ is an open cover of the S -scheme X , then the canonical diagram below should be an equalizer

$$F(X) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$$

Defining $\mathrm{Zar}(S)$

Functor of Points

Let X be an S -scheme. Its functor of points mapping an S -scheme T to $\mathrm{Hom}_S(T, X)$ satisfies the previous conditions, and thus is an object of $\mathrm{Zar}(S)$, which we'll denote $\underline{X}$.

Defining $\mathrm{Zar}(S)$

Affine Line over S $\underline{\mathbb{A}}_S^1$

An important object in $\mathrm{Zar}(S)$ is the **affine line over S object** $\underline{\mathbb{A}}_S^1$, i.e. the functor of points of the S -scheme

$$\mathbb{A}_S^1 := S \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{Z}[X],$$

which sends an S -scheme T to

$$\underline{\mathbb{A}}_S^1(T) = \mathrm{Hom}_S(T, \mathbb{A}_S^1) \cong \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z})}(T, \mathbb{Z}[X]) \cong \Gamma(T, \mathcal{O}_T).$$

Working Inside $\mathrm{Zar}(S)$

Question

Why is $\mathrm{Zar}(S)$ an interesting category to work with?

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Question

Why is $\mathrm{Zar}(S)$ an interesting category to work with?

Answer

It is a sheaf topos, and thus

- (i) It looks very similar to $\mathbf{Set}$,
- (ii) It behaves well under change of base.

Thus $\mathrm{Zar}(S)$ is a good candidate for being our non-standard universe.

Working Inside $\text{Zar}(S)$

$\text{Zar}(S)$ being a topos means that it has an **internal logic**, i.e. a formal device to pretend that its objects are sets.

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Surjections are Epic

For instance, let T be an S -scheme.

If $\alpha : X \rightarrow Y$ is a morphism of S -schemes, we may write internally

$$T \models \forall y : Y. \exists x : X. \alpha(x) = y,$$

which will hold if and only if α is an epimorphism.

Kripke-Joyal Semantics for $\text{Zar}(S)$

The interpretation of arbitrary sentences φ in $\text{Zar}(S)$ is defined through the **Kripke-Joyal semantics**.

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$\top$ and $\perp$

We interpret true and false as:

$$T \models \top : \iff T = T,$$

$$T \models \perp : \iff T = \emptyset.$$

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$$\varphi \implies \psi$$

We interpret $T \models \varphi \implies \psi$ as

For all morphisms $T' \rightarrow T$ in Sch/S , $T' \models \varphi$ implies $T' \models \psi$.

Kripke-Joyal Semantics for $\text{Zar}(S)$

The interpretation of arbitrary sentences φ in $\text{Zar}(S)$ is defined through the **Kripke-Joyal semantics**.

$$\forall s : F. \varphi(s)$$

We interpret $T \models \forall s : F. \varphi(s)$ as

For all morphisms $T' \rightarrow T$ in Sch/S and sections $s \in \Gamma(T', F)$, $T' \models \varphi(s)$.

Kripke-Joyal Semantics for $\text{Zar}(S)$

Proposition

Let T be an S -scheme and φ a formula over T .

- (i) If $T \models \varphi$ and there is an **intuitionistic** proof that $\varphi \implies \psi$, then $T \models \psi$,
- (ii) Let $T' \rightarrow T$ be a morphism of schemes. If $T \models \varphi$, then $T' \models \varphi$,
- (iii) If $T = \bigcup_{i \in I} T_i$ is an open covering and for all i , $T_i \models \varphi$, then $T \models \varphi$.

Intuitionistic Concerns

Internal Reasoning

We can prove statements in an entirely internal way, **provided our proofs are intuitionistically valid.**

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Question

What does this mean in practice?

Intuitionistic Concerns

Although the logical connectives are the same, their *meaning* will be different:

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Proving $\varphi \vee \psi$

To prove $\varphi \vee \psi$, we need to also show whether φ or ψ holds.

Intuitionistic Concerns

No Law of Excluded Middle or Double Negation Exclusion

The following are generally false in $\text{Zar}(S)$:

$$\varphi \vee \neg\varphi \quad \neg\neg\varphi \implies \varphi$$

Moreover, $\text{AC} \implies \text{LEM}$, so we cannot use it.

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Triple Negation Exclusion

But $\neg\neg\neg\varphi \implies \neg\varphi$ holds.

A First Property of $\underline{\mathbb{A}}_S^1$

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Each $\underline{\mathbb{A}}_S^1(T)$ will generally never satisfy a field condition.

But $\underline{\mathbb{A}}_S^1$ as an object of $\text{Zar}(S)$ does!

A First Property of $\underline{\mathbb{A}}_S^1$

$\underline{\mathbb{A}}_S^1$ is a Field

Internally to $\text{Zar}(S)$, the ring $\underline{\mathbb{A}}_S^1$ is a field, in the sense that

$$S \models \forall f : \underline{\mathbb{A}}_S^1. \neg(f = 0) \implies [f \text{ inv.}].$$

A First Property of $\underline{\mathbb{A}}_S^1$

Proof.

By the Kripke-Joyal semantics, we need to show that for every S -scheme T and section $f \in \Gamma(T, \mathcal{O}_T)$, $T \models \neg(f = 0)$, implies $T \models [f \text{ inv.}]$.



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Since $\neg(f = 0)$ means $f = 0 \implies \perp$, we assume that for any T -scheme T' , if the pullback of f to T' is zero, then T' is the empty scheme.



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Since f vanishes on $T' := V(f)$, T' is empty, and so its complement is all of T .



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Since f vanishes on $T' := V(f)$, T' is empty, and so its complement is all of T .

We can now intuitionistically conclude that f is invertible in T . □

Quasi-coherence of $\underline{\mathbb{A}}_S^1$

Definition (Quasi-coherence)

Let $(X, \mathcal{O}_X)$ be a ringed space. We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$ -modules is **quasi-coherent** if for all points $x \in X$, there exists an open nbhd $x \in U \subseteq X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus_{i \in I} \mathcal{O}_U$$

Quasi-coherence of $\underline{\mathbb{A}}_S^1$

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Definition (Synthetic Spec)

Let R be a local ring. The **synthetic spectrum of an R -algebra A** is

$$\mathrm{Spec}(A) := \mathrm{Hom}_{R\text{-Alg}}(A, R),$$

the set of R -algebra homomorphisms from A to R .

Quasi-coherence of $\underline{\mathbb{A}}_S^1$

$\mathrm{Spec}(R)$

The synthetic spectrum of R is the singleton set $\{\star\}$.

Quasi-coherence of $\underline{\mathbb{A}}_S^1$ $\mathrm{Spec}(R)$

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 $\mathrm{Spec}(R/(f))$

The synthetic spectrum of $R/(f)$ is $[[f = 0]]$, i.e. the sub-singleton set

$$\{\star | f = 0\}.$$

Quasi-coherence of $\underline{\mathbb{A}}_S^1$

$$\mathrm{Spec}(R[X_1, \dots, X_n]/(f_1, \dots, f_m))$$

The synthetic spectrum of $R[X_1, \dots, X_n]/(f_1, \dots, f_m)$ is the solution set

$$\{x : R^n \mid f_1(x) = \dots = f_m(x)\}.$$

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$$\{x : R^n \mid f_1(x) = \dots = f_m(x)\}.$$

$$\mathrm{Spec}(R[f^{-1}])$$

The synthetic spectrum of $R[f^{-1}]$ is $[[[f \text{ inv.}]]]$.

Quasi-coherence of $\underline{\mathbb{A}}_S^1$

Definition (Synthetic Quasi-coherence)

An R -module E is **synthetically quasi-coherent** if, for any finitely presented R -algebra A , the canonical R -module homomorphism

$$\begin{aligned} E \otimes_R A &\longrightarrow \operatorname{Hom}(\operatorname{Spec}(A), E) \\ x \otimes f &\longmapsto (\varphi \mapsto \varphi(f)x), \end{aligned}$$

is a bijection.

Quasi-coherence of $\underline{\mathbb{A}}_S^1$

Warning

This is meaningful only in intuitionistic context, otherwise not even R can be synthetically quasi-coherent!

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Lemma ([Ble21, 18.19])

$\underline{\mathbb{A}}_S^1$ is synthetically quasi-coherent as an $\underline{\mathbb{A}}_S^1$ -module.

Using Synthetic Quasi-coherence

Proof of $\underline{\mathbb{A}}_S^1$ is a Field, Revisited.

Let $f : \underline{\mathbb{A}}_S^1$ be such that $\neg(f = 0)$ and consider the f.p. $\underline{\mathbb{A}}_S^1$ -algebra $A := \underline{\mathbb{A}}_S^1/(f)$.



Using Synthetic Quasi-coherence

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Let $f : \underline{\mathbb{A}}_S^1$ be such that $\neg(f = 0)$ and consider the f.p. $\underline{\mathbb{A}}_S^1$ -algebra $A := \underline{\mathbb{A}}_S^1/(f)$. Since $\mathrm{Spec}(A) \cong [[f = 0]] = [[\perp]] = \emptyset$, synthetic quasi-coherence gives a bijection

$$\underline{\mathbb{A}}_S^1/(x) \longleftrightarrow \mathrm{Hom}(\emptyset, \underline{\mathbb{A}}_S^1) = \emptyset.$$

Thus $\underline{\mathbb{A}}_S^1/(f)$ is the zero algebra, and $1 \in (f)$. Hence f is invertible. □

Using Synthetic Quasi-coherence

$\underline{\mathbb{A}}_S^1$ is not Reduced

$\underline{\mathbb{A}}_S^1$ is not reduced:

$$S \models \neg(\forall f : \underline{\mathbb{A}}_S^1. (\bigvee_{n \geq 0} f^n = 0) \implies f = 0).$$

Using Synthetic Quasi-coherence

Proof.

Assume $\underline{\mathbb{A}}_S^1$ is reduced. Then the set $\Delta := \{\varepsilon : \underline{\mathbb{A}}_S^1 | \varepsilon^2 = 0\}$ is equal to $\{0\}$.



Using Synthetic Quasi-coherence

Proof.

Assume $\underline{\mathbb{A}}_S^1$ is reduced. Then the set $\Delta := \{\varepsilon : \underline{\mathbb{A}}_S^1 | \varepsilon^2 = 0\}$ is equal to $\{0\}$.
By synthetic quasi-coherence, we get an isomorphism

$$\underline{\mathbb{A}}_S^1[T]/(T^2) \longleftrightarrow \mathrm{Hom}(\underline{\mathbb{A}}_S^1[T]/(T^2), \underline{\mathbb{A}}_S^1) \cong \mathrm{Hom}(\Delta, \underline{\mathbb{A}}_S^1) = \{0\}.$$



Using Synthetic Quasi-coherence

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$$\underline{\mathbb{A}}_S^1[T]/(T^2) \longleftrightarrow \mathrm{Hom}(\underline{\mathbb{A}}_S^1[T]/(T^2), \underline{\mathbb{A}}_S^1) \cong \mathrm{Hom}(\Delta, \underline{\mathbb{A}}_S^1) = \{0\}.$$

Hence $[T]$ gets mapped to zero, and thus $T \in (T^2)$. Therefore $1 = 0$ in $\underline{\mathbb{A}}_S^1$, a contradiction. □

Using Synthetic Quasi-coherence

Non-Reduced Fields

Classically, there are no non-reduced fields!

Using Synthetic Quasi-coherence

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But being reduced is not stable under base change, and all internal statements in $\text{Zar}(S)$ are.

Using Synthetic Quasi-coherence

Non-Reduced Fields

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But being reduced is not stable under base change, and all internal statements in $\mathrm{Zar}(S)$ are.

Question

Are there really non-zero nilpotent elements in $\underline{\mathbb{A}}_S^1$?

Using Synthetic Quasi-coherence

Facts about Nilpotents in $\underline{\mathbb{A}}_S^1$

Let $f : \underline{\mathbb{A}}_S^1$. The following are equivalent:

- (i) f is not invertible,
- (ii) f is nilpotent,
- (iii) f is not not zero.

Using Synthetic Quasi-coherence

All Functions are Polynomials

Every function $\underline{\mathbb{A}}_S^1 \rightarrow \underline{\mathbb{A}}_S^1$ is given by a unique polynomial in $\underline{\mathbb{A}}_S^1[T]$.

Using Synthetic Quasi-coherence

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Proof.

Since $\mathrm{Spec}(\underline{\mathbb{A}}_S^1[T]) \cong \underline{\mathbb{A}}_S^1$, the claim is an immediate consequence of quasi-coherence. □

Using Synthetic Quasi-coherence

Classically Impossible

This is too cannot happen classically.

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For finite fields K , uniqueness fails, since the zero map is given by both

$$x \longmapsto 0,$$

$$x \longmapsto \prod_{\alpha \in K} (x - \alpha).$$

Using Synthetic Quasi-coherence

Classically Impossible

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For finite fields K , uniqueness fails, since the zero map is given by both

$$\begin{aligned}x &\longmapsto 0, \\x &\longmapsto \prod_{\alpha \in K} (x - \alpha).\end{aligned}$$

For infinite fields K , existence fails since $|K[T]| = |K| < |\mathrm{Hom}(K, K)|$.

References I

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- [SA20] Jonathan Sterling and Carlo Angiuli, *Gluing Models of Type Theory along Flat Functors* (January 2020), 35 (en).