

UNIVERSITY OF
CAMBRIDGE

MATHEMATICS TRIPOS

Part III

Coxeter Groups

Lent, 2022

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Figure 1: Escher's *Circle Limit*

0 Introduction

Coxeter groups are ubiquitous objects in mathematics, with many connections to algebra, geometry, topology, and representation theory. They are groups generated by reflections satisfying specific relations, and in some cases these groups can be interpreted as reflections across hyperplanes in Euclidean, spherical or hyperbolic space. Dihedral groups (reflection groups of polygons), and symmetric groups (reflection groups of standard simplices) are two examples of families of spherical Coxeter groups.

This course will serve primarily as an introduction to Coxeter groups, and their rich and elegant theory. Along the way, we will encounter tools utilised across many different fields of mathematics. As a whole the course will be primarily algebraic but the tools come from the following fields: algebra, geometric group theory, combinatorics, geometry and topology. The study of Coxeter groups via these five fields can be loosely associated to the five points in the synopsis below.

Course synopsis:

1. Coxeter systems: definition, examples, Coxeter diagrams.
2. The word problem: Cayley graphs, the exchange condition, Tits' solution to the word problem for Coxeter groups.
3. Combinatorial theory of Coxeter groups: special subgroups, parabolic subgroups, cosets and elements of shortest and longest length.
4. The Tits representation, and classification of finite and affine Coxeter groups: spherical, Euclidean and hyperbolic.
5. The Davis complex and its properties: as a basic construction, as the geometric realisation of a poset, as a CW complex, contractibility, as a classifying space for a family of subgroups.

At the end of the course we may spend some time on one or more of the following related topics: Artin groups, buildings, Weyl groups, cube complexes.

References are [Bro89, Dav08, Hum90], but main reference is [Dav08], for which there is e-book online. Auxiliary book is not in this list.

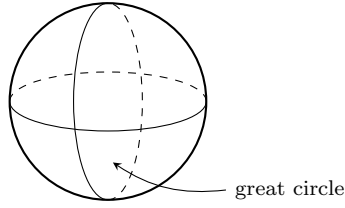


Figure 2: $\mathbb{S}^2$

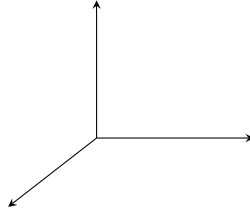


Figure 3: $\mathbb{E}^3$

1 Geometric Reflection Groups

Lots of combinatorial group theory, but need no previous exposure to combinatorics.

Some Riemannian geometry is needed for the following. This is because Coxeter groups are discrete groups generated by 'reflections', and we need to make that notion precise (which will happen in Sec. 2 and 3).

This section mostly concerns motivating examples but we will also introduce a Theorem proved in Sec. 6.

Recall the following notions:

Definition 1.1 (Riemannian manifold). A **Riemannian manifold** is a smooth manifold M equipped with a positive definite inner product on $T_x M \forall x \in M$.

This inner product allows us to define isometries:

Definition 1.2 (Isometry). An **isometry** on M is a diffeomorphism $M \xrightarrow{\sim} M$ that preserves the inner product at every point.

We can define a **metric** on M and thus **geodesics**, which are locally distance minimising between two points. We can also define sectional curvature of M .

Throughout the course we will refer to three main examples of Riemannian manifolds, which we'll now introduce.

1.0.1 Notation

Some notation remarks:

- $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the n -dim sphere centred at origin with round metric.
- $\mathbb{E}^n \cong (\mathbb{R}^d, \cdot)$ Euclidean space.
- $\mathbb{H}^n$ is real hyperbolic space in Figure 4 and 5.
- We'll write $\mathbb{X}^n$ for one of $\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n$.

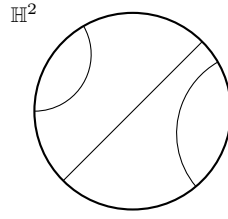


Figure 4: $\mathbb{H}^2$

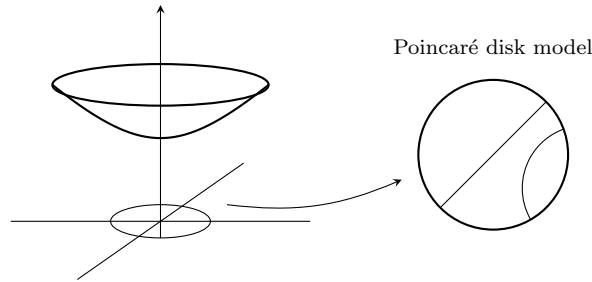


Figure 5: Poincaré disk model.

- The isometry group of $\mathbb{X}^n$ is denoted $\text{Isom}(\mathbb{X}^n)$.
- The order of composition is given as if $s_1, s_2 \in \text{Isom}(\mathbb{X}^n)$, then $s_1 s_2 \cdot x = s_1(s_2(x))$.

Remark. Order of composition has been clarified at Lecture 03. Previous Lectures may use inconsistent notation.

Remark. These three spaces are all the Riemannian manifolds Riemannian constant sectional curvature, respectively, 1,0,-1.

This gives them existence of nice properties, such as existence of hyperplanes.

The hardest to visualise is by far $\mathbb{H}^n$. In the case of $n = 2$, we can draw Poincaré disk model.

We can think of $\mathbb{H}^n$ as a solution to the equation $x^2 + y^2 - z^2 = -1$ for $z > 0$, which is an infinite space. In the Poincaré disk model, this is drawn as a unit disk, with all the points lie inside.

We have a mapping from the coordinates in the previous equation to the Poincaré disk model given by taking straight lines between the point $(0, -1, 0)$ and a given point, and projecting down. The geodesics map onto the unit disk either as diameters or as curves that start and end on the unit disk at right angles.

A nice visualisation of the hyperbolic plane is on the book cover. Note that all those pentagons are of the same area, as we're mapping an infinite space in finite area.

For the other manifolds, in $\mathbb{R}^n$ geodesics are straight lines, whereas in $\mathbb{S}^n$ they're the great circles.

Definition 1.3 (Hyperplane). We say a subspace of $\mathbb{X}^n$ is **totally geodesics** if all the geodesics between any two points in it lie inside it.

A **hyperplane** $\mathcal{H} \subset \mathbb{X}^n$ is a totally geodesic, codim-1 submanifold of $\mathbb{X}^n$.

A hyperplane $\mathcal{H}$ separates $\mathbb{X}^n$ into two connected components, called the half

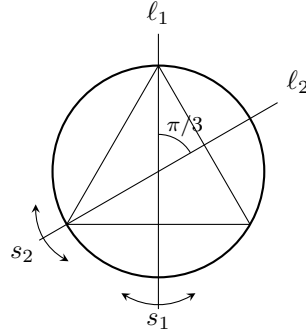


Figure 6: D_6

| spaces.

| **Remark.** This only works for the three $\mathbb{X}^n$ that we chose, it does not hold in generality.

| **Question.** Are diameters in the disk model not hyperplanes?

| **Theorem 1.4.** For each hyperplane $\mathcal{H} \subseteq \mathbb{X}^n$ there exists a reflection in $\text{Isom}(X^n)$ which

- (i) Fixes the H ,
- (ii) Exchanges the associated half-spaces.

Proof. Will do in later lectures. □

We will arrange these reflections in a way that they generate a group, which is going to be our first example of Coxeter group.

| **Remark.** The first two examples are not that great because although they are geometric reflection groups, they don't really move around hyperplanes.

| **Example ($\mathbb{S}^1$).** Take $\mathbb{S}^1$ with "hyperplanes" d_1 and d_2 meeting at angle $\frac{\pi}{m}$, and s_i reflection across l_i (see Figure 6). Then $s_1 s_2$ is rotation by $\frac{2\pi}{m}$, i.e. $\langle s_1, s_2 \rangle \cong C_m$.

The group $W \cong D_{2m}$ defined with relations

$$W = \langle s_1, s_2 | s_1^2 = s_2^2 = (s_1 s_2)^m = e \rangle,$$

which we know is a presentation for D_{2m} . As sanity check we can inscribe a triangle inside circle as in Figure 6.

| **Definition 1.5** (Dihedral group). D_{2m} is the dihedral group of order $2m$, which is the group of symmetries of the m -gon.

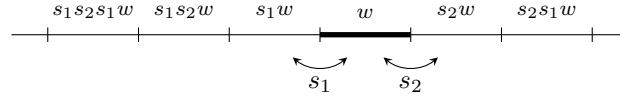


Figure 7: Fundamental Domain on D_∞ showing the action $D_\infty \curvearrowright \mathbb{R}$.

Example (Infinite dihedral group). Take $\mathbb{E}^1 = \mathbb{R}^1$, in which the hyperplanes are points, take 0, 1, and s_1, s_2 reflections about these points, i.e. $\forall t \in \mathbb{R}$

$$\begin{aligned} s_1(t) &= -t \\ s_2(t) &= 2 - t. \end{aligned}$$

Since s_1s_2 is the translation by 2, i.e. $\langle s_1, s_2 \rangle \cong \mathbb{Z}$, hence $W = \langle s_1, s_2 | s_1^2 = s_2^2 = e \rangle = D_\infty$ is the infinite dihedral group.

Remark. If $\langle s_1, s_2 \rangle \cong \mathbb{Z}$ i.e. s_1s_2 has infinite order we may write $(s_1s_2)^\infty = e$.

This allows us to have more uniform presentations in the general case. For example, our previous presentation becomes $W = \langle s_1, s_2 | s_1^2 = s_2^2 = (s_1s_2)^\infty = e \rangle$.

The following definition is crucial:

Definition 1.6 (Fundamental domain). Let X be a topological space and $G \curvearrowright X$ by homeomorphism and G_x be the orbit of $x \in X$.

A **fundamental domain** for the action $G \curvearrowright X$, is a subset $K \subset X$ such that the following hold

- K is closed and connected,
- $\forall x \in X$ we have $G_x \cap K \neq \emptyset$,
- $\forall x \in \overset{\circ}{K}$ we have $G_x \cap K = \{x\}$.

Definition 1.7 (Strict fundamental domain). A fundamental domain K is **strict** if $\forall x \in K$ we have $G_x \cap K = x$, i.e. K contains exactly one point from each orbit.

Example (D_∞). See Figure 7. The closed interval $[0, 1]$ is a strict fundamental domain and so is any interval $[t, t + 1]$ for $t \in \mathbb{R}$, whereas $[t, t + 2]$ is a fundamental domain but is not strict.

Lecture 2

Remark. Fundamental domains give rise to tilings or tessellations of $\mathbb{X}^n$.

Definition 1.8 (Simplex). A **simplex** $\sigma_k \subseteq \mathbb{X}^n$ for $n \geq k$ is the convex hull of $k + 1$ basis vectors in $\mathbb{X}^n$ (it is k -dimensional). We say a simplex is **regular** if all edges have the same length. Δ^k is the **regular Euclidean k -simplex**.

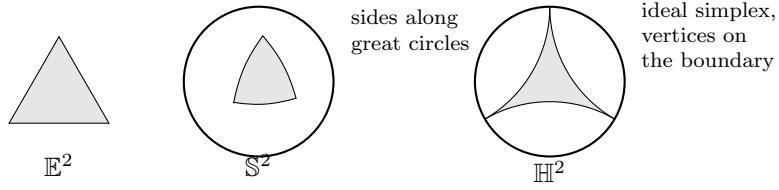


Figure 8: Simplices for various $\mathbb{X}^2$.

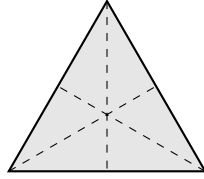


Figure 9: Δ^2 .

Remark. This definition has been update to be closer to the one given in Algebraic Topology. The previous definition is reported here for completeness sake:

Definition 1.9 (Simplex (old defn.)). A **simplex** D^n is the convex hull of $n + 1$ points. We say a simplex is **regular** if any perm of vertices can be realised by isometry of $\mathbb{X}^n$ (see Figure 8).

Remark. Isometrics preserve distances, so every new point must be at same distance to others.

This lecture will be mostly concerned about spherical simplices.

Definition 1.10 (Complex Polytope). A **complex polytope** $P \subseteq \mathbb{X}^n$ is a convex, compact intersection of a finite number of closed half-spaces in $\mathbb{X}^n$ with non-empty interior.

Definition 1.11 (Link (informal)). The **link** of a vertex v in P is $\text{link}_P(v) = P \cap n - 1\text{-sphere centred at } v$ (sphere in $T_v\mathbb{X}^n$). This is a spherical $(n - 1)\text{-dim}$ polytope.

Definition 1.12 (Simple Polytope). P is **simple** if $\forall v \in P$, $\text{link}_P(v)$ is a spherical simplex.

Example ($n = 2$). Then a convex polytope $P \subseteq \mathbb{X}^2$ is a convex polygon and every P is simple.

Remark. For any polytope $P \subseteq \mathbb{X}^n$, the faces of P lie in hyperplanes.

Theorem 1.13 (Geometric Reflection Groups Coxeter). Let $P \subseteq \mathbb{X}^n$ a simple convex polytope with $n \geq 2$, and $\{F_i\}_{i \in I}$ finite set of codim-1 faces of P . Then each F_i lies in $\mathcal{H}_i \subset \mathbb{X}^n$.

Suppose $\forall i \neq j$ if $F_i \cap F_j \neq \emptyset$ then $\mathcal{H}_i$ and $\mathcal{H}_j$ intersect at an angle of $\frac{\pi}{m_{ij}}$ where $m_{ij} \in \mathbb{Z}_{\geq 2}$. Set $m_{ii} = 1$ and $m_{ij} = \infty$ when $F_i \cap F_j = \emptyset$ and let s_i be reflection across $\mathcal{H}_i$ in $\text{Isom}(\mathbb{X}^n)$.

Let W be the group generated by $\{s_i\}_{i \in I}$. Then

- (i) W has presentation $W = \langle s_i \mid (s_i s_j)^{m_{ij}} = e \forall i, j \in I \rangle$.
- (ii) W is a discrete subgroup of $\text{Isom}(\mathbb{X}^n)$ (enforced by fixed angles).
- (iii) P is a strict fundamental domain for $W \curvearrowright \mathbb{X}^n$ and the action induces a tessellation of $\mathbb{X}^n$ by copies of P .

Remark. Setting $m_{ii} = 1$ gives $\forall i \in I : (s_i s_i)^1 = s_i^2 = e$.

Definition 1.14 (Geometric reflection group). A group W is a **geometric reflection group** if it is D_{2m}, D_∞ or a group satisfying the conditions of the previous Theorem.

Remark. It is in general quite hard to tell whether a given Coxeter group arises as a geometric reflection group, without explicitly having the polytope.

Definition 1.15. We say W is **spherical** if $\mathbb{X} = \mathbb{S}^n$, **Euclidean** if $\mathbb{X}^n = \mathbb{E}^n$ or **hyperbolic** if $\mathbb{X}^n = \mathbb{H}^n$.

Remark. Coxeter had a correspondence with Escher, apparently he really really liked his polytopes.

Remark. Geometric reflection groups are our first examples of Coxeter groups. Coxeter classified all spherical and Euclidean groups. The hyperbolic case is still open, at least for $n \geq 3$.

1.1 Examples of geometric reflection groups

Some examples will be on ES#1.

Example (Triangle groups). $\forall p, q, r \in \mathbb{Z}_{\geq 2}$ with $p \leq q \leq r$ we have a triangle $P \subseteq \mathbb{X}^2$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, although which $\mathbb{X}^2$ it is will depend on the angles.

Then $W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^p = (s_2 s_3)^q = (s_3 s_1)^r = e \rangle$,

- When $\mathbb{X}^2 = \mathbb{S}^2$, then $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} > \pi$, possible triples: $(2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)$.
- When $\mathbb{X}^2 = \mathbb{E}^2$, then $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi$, finite possibilities, e.g. $p = q = r = 3$.
- When $\mathbb{X}^2 = \mathbb{H}^2$, then $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$, infinite possibilities.

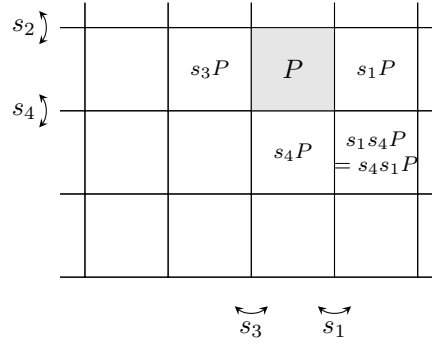


Figure 10: Tiling of $\mathbb{E}^2$ by 2-cubes.

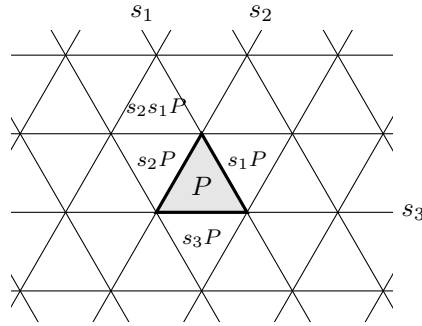


Figure 11: Triangle group $(3,3,3)$.

Example (Symmetric group). For $n \geq 2$, and P regular Euclidean simplex $\Delta^n \subseteq \mathbb{E}^n$.

Labelling vertices in the set $\{1, \dots, n+1\}$,

Then $\text{Isom}(\Delta^n) \cong S_{n+1}$ the symmetric group of $n+1$ letters. By embedding $\Delta^n \subseteq \mathbb{E}^n$ such that vertices lie on $\mathbb{S}^{n-1}$ and then *puff out* D^n to lie on $\mathbb{S}^{n-1}$, we get a tessellation of $\mathbb{S}^{n-1}$ by $\partial\Delta^n$. Taking barycentric subdivision of $\partial\Delta^n$ gives us a finer subdivision.

This gives

$$W = S_{n+1} = \langle s_1, \dots, s_n \mid s_i^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = e \rangle,$$

where indices in first case are $i \in I$, in the second $|i-j| \geq 2$ and the third $1 \leq i \leq n$.

By considering the transpositions $s_i = (i, i+1)$ we observe that S_{n+1} is the permutation group.

Lecture 3

Example (Tiling of $\mathbb{E}^n$ by n -cubes). Observe in Figure 10 that a copy of D_∞ lives in every row.

Example (Triangle group $(3,3,3)$). We have three dihedral groups as depicted in Figures 11 and 12. It is an infinite group that does not contain D_∞ .

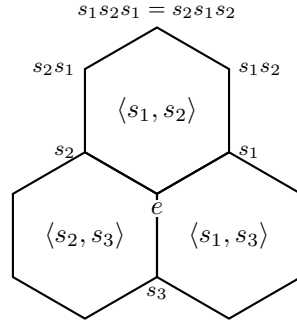


Figure 12: Three copies of D_6 live as duals to the triangle group $(3, 3, 3)$.

2 Abstract Reflection Groups

2.1 Coxeter Systems Formalism

This section aims to prove two abstractions of geometric reflection groups and to prove their equivalence.

Definition 2.1 (Tits 50s). Let $S = \{s_i\}_{i \in I}$ with I finite indexing. A **Coxeter matrix** is a symmetric $I \times I$ matrix, $m = \{m_{i,j}\}_{i,j \in I}$, such that

- (i) $\forall i \in I : m_{i,i} = 1$,
- (ii) $\forall i \neq j \in I : m_{i,j} = m_{j,i} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.

A **Coxeter group** W associated with m is a group presented as

$$W = \langle S \mid (s_i s_j)^{m_{i,j}} = e \forall i, j \in I \rangle,$$

and the pair (W, S) is a **Coxeter system**.

Remark. Note that $\forall s_i \in S : m_{i,i} = 1 \implies s_i^2 = e$. Also $(s_i s_j)^{m_{i,j}}$ can be rewritten as

$$s_i s_j s_i s_j \cdots = s_j s_i s_j s_i \cdots$$

where on both sides there are $m_{i,j}$ generators.

Remark. Geometric reflection groups are Coxeter groups by Theorem 1.13.

Warning. Not all Coxeter groups are geometric reflection groups!

Cannot always find a polytope $P \subseteq \mathbb{X}^n$ that gives W .

Remark. A Coxeter group W can correspond to multiple Coxeter systems. See ES#1.

Interestingly there are moves on Coxeter systems that give isomorphic Coxeter groups.

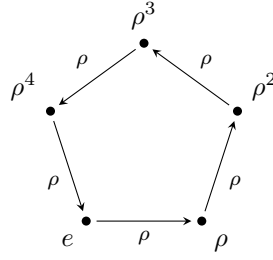


Figure 13: $\text{Cay}_g(C_5)$

Conjecture (Isomorphism Problem for Coxeter groups [Bah05]). This set of moves is exhaustive, i.e. there are no other moves that give rise to isomorphic Coxeter groups.

Remark. One can define (W, S) with S infinite, but for those the theory is a bit different. Thus in the following S will be finite.

Next result is a consequence of the Tits representation, but we need it now

Corollary 2.2. Let (W, S) be a Coxeter system. Then the elements of S are pairwise distinct and for all $i, j \in I$, $s_i s_j$ has order $m_{i,j}$.

Remark. See discussion of presentations in Infinite Groups for why this is non-trivial.

2.2 Combinatorics of Coxeter groups

In the following let G be a group with a generating set $e \notin S$.

Definition 2.3. The **Cayley graph** of G w.r.t. to S , denoted $\text{Cay}_S(G)$ is the graph with vertex set G .

It has directed edge set

$$\{(g, gs) | g \in G, s \in S, s^2 \neq e\},$$

and undirected edge set

$$\{\{g, gs\} | g \in G, s \in S, s^2 \neq e\}.$$

All edges are labelled by corresponding $s \in S$.

Example. Let $G = C_5 = \langle g \rangle$, then $\text{Cay}_g(G)$ is as in Figure 13.

Exercise. Draw some $\text{Cay}_X(F(X))$ for some small $|X|$.

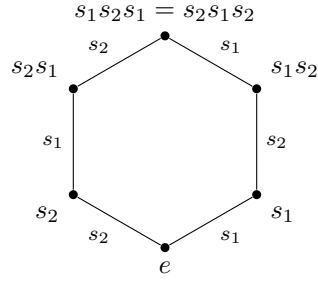


Figure 14: $\text{Cay}(D_6)$

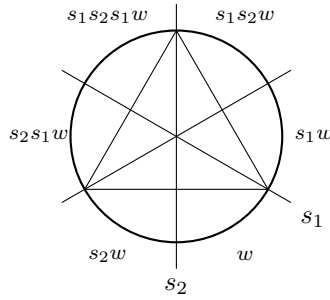


Figure 15: $D_6 \rtimes S^1$.

Remark. In our examples S is always a set of involutions, so all edges in $\text{Cay}_S(G)$ are undirected.

Remark. Since S generates G , $\text{Cay}_S(G)$ is always connected.

From Cor 2.2, we also know that for (W, S) Coxeter system, $\text{Cay}_S(W)$ is **simple**, i.e. with no loops at v and no double edges.

Lemma 2.4. G acts on $\text{Cay}_S(G)$ via multiplication on the left. This action preserves edge labels. Under this action, if $s^2 = e$, then $gs g^{-1}$ is the unique group element which flips the undirected edge $\{g, gs\}$.

Proof. On ES#1, just check that it is an action and that it flips. □

Exercise (D_{2m}). Take D_6 , have Cay as in Figure 14. Note that it can be inscribed in Figure 15, a re-visitation of Figure 6.

Remark. In the case of geometric reflection groups, $\text{Cay}_S(W)$ is dual to the tessellation of $\mathbb{X}^n$ by the polytope P . See Figure 11 for the triangle group $(3, 3, 3)$ and Figure 10 for n -cubes in $\mathbb{E}^n$.

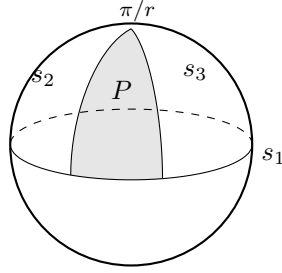


Figure 16: Triangle group $(2, 2, r)$.

Remark. If $m_{i,j}$ is finite, will see a copy of $2m_{i,j}$ -gon in the Cayley graph. This is why drawings are nice, as we have nice $m_{i,j}$.

Example (Triangle group $(2, 2, r)$). When have tessellation of S^n always have finite group, as can only tile sphere with finite number of triangles, as in Figure 16.

For Euclidean reflection groups, the opposite is true: cannot be tiled with finite number of elements.

2.3 Reflection System Formalism

Will prove equivalent to Coxeter system (W, S) formalism.

Remark. All our generating sets from now on will be involutions.

Definition 2.5 (Word). Given a group G , with generating set S of involution. An **element** is a product of generators

$$s_1 \cdots s_n,$$

$s_i \in S$, whereas a **word** is a finite sequence of generators

$$(s_1, \dots, s_n),$$

with $s_i \in S$. The **word length** of a group element w.r.t. a generating set S is

$$l_S(g) = \min\{n \in \mathbb{N} | g = s_1 \cdots s_n, s_i \in S\},$$

and $l_S(e) = 0$.

If $l_S(g) = h \geq 1$ and $g = s_1 \cdots s_n$, then $(s_1, \dots, s_n)$ is called a **reduced word** for g .

Example (D_6). In D_6 , have $s_1 s_2 s_1 = s_2 s_1 s_2$ an element and (s_1, s_2, s_1) and (s_2, s_1, s_2) are reduced words for this element.

Definition 2.6 (Word metric). The **word metric** on G is given by

$$d_S(g, h) = l_S(g^{-1}h),$$

for $g, h \in G$. This extends to a path metric on $\text{Cay}_S(G, S)$, where each edge is given

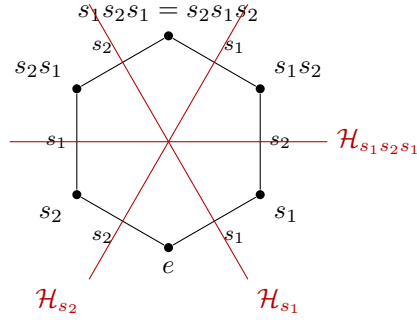


Figure 17: Pre-reflection system for D_6 .

length 1 and the distance between any two vertices is the shortest path between them.

Example (D_6). On D_6 , we have

$$d_{\{s_1, s_2\}}(e, s_1s_2) = l_S(s_1s_2) = 2.$$

Example. If $d_S(e, g) = l_S(g) = k$ and if $(s_1, \dots, s_k)$ is a reduced word for g , we have a path of minimal length k from e to g in $\text{Cay}_S(G)$.

Definition 2.7 (Pre-reflection system). A **pre-reflection system** for a group G is a pair (X, R) such that

- (i) X is a connected simple graph.
- (ii) $G \curvearrowright X$ by graph automorphisms.
- (iii) R is a subset of G and moreover.

and moreover

- (a) Every $r \in R$ is an involution
- (b) R is closed under conjugation, i.e. $\forall g \in G, r \in R : grg^{-1} \in R$
- (c) R generates G
- (d) $\forall \{v, w\} \in E(X) \exists ! r \in R$ which flips $\{v, w\}$, i.e. it interchanges v and w .
- (e) Each $r \in R$ flips at least one edge.

For $r \in R$, let H_r be the set of midpoints of edges flipped by r .

Example (D_6). Take $X = \text{Cay}_S(W)$ but R has to be closed under conjugation, so we take

$$R = \{s_1, s_2, s_1s_2s_1 = s_2s_1s_2\},$$

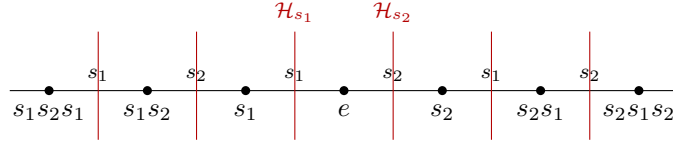


Figure 18: Pre-reflection system for D_∞ .

as in Figure 17. Then (X, R) a pre-reflection system for $W = D_6$.

Example (D_∞). Take $W = D_\infty$, $X = \text{Cay}_S(W)$ and

$$R = \{ws_iw^{-1} | s_i \in S, w \in W\},$$

as in Figure 18. Thus (X, R) is a pre-reflection system for D_∞ . So even though generating sets are always finite, R does not need to be.

Question. Is there a notion of morphism of pre-reflection systems?

Lemma 2.8. If (X, R) is a pre-reflection system for G , then G acts transitively on $V(X)$.

Proof. Since X is connected, there exists a path between any two vertices v and w

$$(v = v_0, v_1, \dots, v_k = w).$$

Let r_1 be the unique involution which flips $\{v_i, v_{i+1}\}$. Then $r_{k-1} \cdot r_k \cdots r_0 v_1 = w$. $\square$

Lemma 2.9. Let (W, S) be a Coxeter system and

$$R = \{ws w^{-1} | s \in S, w \in W\},$$

then $(\text{Cay}_S(W), R)$ is a pre-reflection system of W .

Proof. By Corollary 2.2 and since S generates W , $\text{Cay}_S(W)$ is always a connected simple graph. Moreover,

$$\forall w s w^{-1} \in R : (w s w^{-1})^2 = e,$$

and this is the unique reflection which flips the edge $\{w, ws\} \in \text{Cay}_S(W)$. $\square$

Definition 2.10 (Reflection system). Let (X, R) be a pre-reflection system for G . Then (X, R) is a **reflection system** if it also satisfied

- (f) For each $r \in R$, $X \setminus H_r$ has exactly two components.

Lecture 5

3 Combinatorics of Coxeter Groups

Dedicated to proving

Theorem 3.1. Let W be a group generated by a set S of distinct involutions.

The following are equivalent:

- (i) (W, S) is a Coxeter system,
- (ii) Let $X = \text{Cay}_S(W)$ and $R = \{ws w^{-1} | s \in S, w \in W\}$. Then (X, R) is a reflection system.
- (iii) (W, S) satisfies the **deletion condition**.
- (iv) (W, S) satisfies the **exchange condition**.

Definition 3.2. The pair (W, S) is said to satisfy **deletion condition** if the following holds:

- (D) If $w = (s_1, \dots, s_k)$ is a word in WS with $l_S(s_1, \dots, s_k) < k$, then there exists indices $i < j$ s.t.

$$s_1 \cdots s_k = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k,$$

where $\widehat{s}$ means delete the letter s .

Definition 3.3. The pair (W, S) is said to satisfy **exchange condition** if the following holds:

- (E) If $w = (s_1, \dots, s_k)$ is a reduced word in W , then for every $s \in S$, either $l_S(ss_1 \cdots s_k) = k + 1$, or

$$w = s_1 \cdots s_k = ss_1 \cdots \widehat{s_i} \cdots s_k,$$

for $1 \leq i \leq k$.

Over the next two lectures will prove (iii) $\implies$ (iv), then some Lemma to show (i) $\implies$ (ii) and (ii) $\implies$ (iii). The final proof (iv) $\implies$ (i) is a consequence of Tits solution to the word problem.

Remark. Exchange s_i for something in the beginning.

Proof. (iii) $\implies$ (iv): Suppose $(s_1, \dots, s_k)$ for $w = s_1 \cdots s_k$ and $s_0 \in S$. Then

$$\begin{aligned} l_S(s_0 s_1 \cdots) &= l_S(s_0 w) \\ &\leq l_S(s_0) + l_S(w) \\ &= R + 1, \end{aligned}$$

by fact proved in ES#1.

So if $l_S(s_0 w) = R + 1$, we're done, hence suppose $l_S(s_0 w) < R + 1$. Then by (D), $\exists 0 \leq i < j \leq R$ such that

$$s_0 w = s_0 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k.$$

Since $(s_1, \dots, s_k)$ is reduced then we must have $i = 0$, otherwise multiplying on the left gives contradiction.

Hence

$$\begin{aligned} s_0 w &= \widehat{s_0} \cdots \widehat{s_j} \cdots s_k \\ &= s_1 \cdots \widehat{s_j} \cdots s_k, \end{aligned}$$

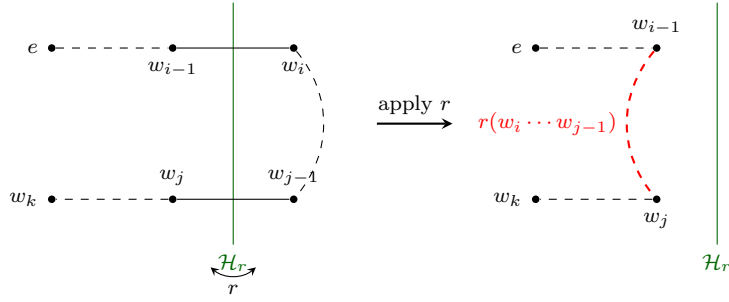


Figure 19: Illustration of idea of Lemma.

thus multiplying on the left by s_0 gives

$$s_0 w = s_0 s_1 \cdots \widehat{s_j} \cdots s_k.$$

Hence (w, s) satisfies (E). $\square$

Remark. There is another equivalent condition, called folding.

Remark. Let W be generated by S as in Theorem 3.1. Then there exists a bijection

$$\{\text{words in } S\} \longleftrightarrow \{\text{paths in } \text{Cay}_s(w) \text{ that start at } e\}$$

Example (2.12). Let $R = \{ws w^{-1} | w \in W, s \in S\}$.

$$e \xrightarrow{s_1} s_1 \xrightarrow{s_2} s_1 s_2 \xrightarrow{\quad \cdots \quad} s_1 \cdots s_k$$

Then from Lemma 2.4, $\exists! r \in R$, which flips the edge $s_1 \cdots s_{j-1}, s_1 \cdots s_j$ given by

$$r_j = s_1 \cdots s_j s_{j-1} \cdots s_1,$$

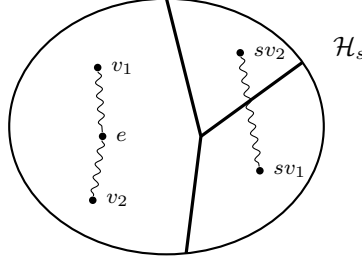
e.g. $r_1 = s_1, r_2 = s_1 s_2 s_1, \dots$ a **reflection sequence** $(r_1, \dots, r_k)$ for a word $(s_1, \dots, s_k)$. Have

$$\mathcal{H}_r = \{\text{midpoint of } \{v, w\} | r \text{ flips } \{v, w\}\},$$

the **wall** of r . We say $(s_1, \dots, s_k)$ **crosses** $\mathcal{H}_r$ if associated path does, i.e. $(s_1, \dots, s_k)$ crosses $\mathcal{H}_{r_1}, \dots, \mathcal{H}_{r_k}$.

Question. What do the walls $\mathcal{H}_r$ look like?

Answer. Removing $\mathcal{H}_r$ from $\text{Cay}_S(W)$ cuts exactly the edges flipped by r , as in Figure 20. We will show that this always leaves at most two components (Lemma 3.5), and that for a Coxeter system it leaves exactly two, swapped by r .


 Figure 20: Walls $\mathcal{H}_r$.

Lemma 3.4. Let W, S, R as above and $(s_1, \dots, s_k)$ a word in S with associated refl. seq $(r_1, \dots, r_k)$ such that $r_i = r_j$ for some $1 \leq i < j \leq k$. Then in W

$$s_1 \cdots s_k = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k.$$

Proof. Let $r_i = r_j$ and $w_p = s_1 \cdots s_p$. Then in $\text{Cay}_S(W)$ we have situation as in Figure 19.

Apply r to $w_i \cdots w_{j-1}$, thus $w \odot \text{Cay}_S(W)$ preserves edge labels so get new path to w_k

$$(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{j-1}, s_{j+1}, \dots, s_k),$$

as required. $\square$

Lemma 3.5. Have W, S, R as above. Then for each $\in R$, $\text{Cay}_S(W) \setminus \mathcal{H}_r$ has at most two connected components.

Proof. Let $r = wsw^{-1}$ for some $w \in W, s \in S$.

We claim that $w \cdots \mathcal{H}_s = \mathcal{H}_{ws w^{-1}} = \mathcal{H}_r$ as sets, so need to prove double inclusion. Can prove by word manipulation or, sketch: s flips edged $\{g, gs'\}$ then $ws w^{-1}$ flips $\{wg, wgs'\} = w \cdot \{g, gs'\}$ (with edge s').

If believe claim then WLOG can prove just for $\mathcal{H}_s$, because $w \odot \text{Cay}_S(W)$ by isometries.

First we show that for all $v \in V(\text{Cay}_S(W)) = W$, then either v or sv is in the same component of $\text{Cay}_S(W) \setminus \mathcal{H}_s$ as e .

Let $(s_1, \dots, s_k)$ be a reduced word for $v \rightsquigarrow$ in $\text{Cay}_S(W)$ from e to v with associated refl. sequence $(r_1, \dots, r_k)$, which tells us exactly which walls this crosses. If $s \neq r_i$, then $\implies e$ and v are in same component of $\text{Cay}_S(W) \setminus \mathcal{H}_s$.

Suppose $s = r_i$ for some i . Then by Lemma 3.4, since $(s_1, \dots, s_k)$ is reduced, $r_j \neq r_i$ for all $j \neq i$. Then the word $(s, s_1, \dots, s_k)$ has reflection sequence $(s, r_1, \dots, r_k)$, where $r'_j = sr_j s$. Thus $r'_i = sss = s$ and $r'_j \neq s, \forall i \neq j$, hence $(s, r'_1, \dots, r'_k)$ as exactly 2 instances of $s \rightsquigarrow$ apply Lemma 3.4 and delete to get a word for sv which $\rightsquigarrow$ path from e to sv not crossing $\mathcal{H}_s$, which implies e and sv are in the same component.

We now claim that this suffices. Why? Suppose that $\text{Cay}_S(W) \setminus \mathcal{H}_s$ has more than two components (see Figure 20). Then $\exists v_1 \neq v_2$ in the same component as e such that sv_1 and sv_2 are in two other components.

Now every path from sv_1 to sv_2 has to cross $\mathcal{H}_s$, thus every path v_1 to v_2 crosses $s \cdot \mathcal{H}_s$, but this is a contradiction. $\square$

Lemma 3.6 (# Wall Crossings for Path). For any word $(s_1, \dots, s_k) = \underline{s}$, let $n(r, \underline{s})$ be the number of times the corresponding path crosses $\mathcal{H}_r$ in $\text{Cay}_S(W)$.

The following hold:

- (i) For any word $(s_1, \dots, s_k) = \underline{s}$, with $w = s_1 \cdots s_k$, then any $r \in R$, $(-1)^{n(r, \underline{s})} \in \{\pm 1\}$ depends only on $w \in W$.
- (ii) $\exists$ hom

$$\begin{aligned} \exists \Phi : W &\longrightarrow \text{Symm}(R \times \{\pm 1\}) \\ w &\longmapsto \varphi_w, \end{aligned}$$

where $\varphi_w(r, \varepsilon) = (wrw^{-1}, (-1)^{n(r, \underline{s})}\varepsilon)$, where $\underline{s}$ is some word for w .

Remark. If take associated reflection system to $\underline{s}$, $n(r, \underline{s})$ is the number of times r shows up.

Lecture 6

Remark. We're filling in the proof of 3.1. Already proved (iii) $\implies$ (iv).

For the final implication we need the following result.

Theorem 3.7 (Tits' Solution to the Word Problem). Let W be generated by a set S of involutions such that (W, S) satisfies the exchange condition, and let m_{st} be the order of st . Then a word in S is reduced if and only if it cannot be shortened by a sequence of the following moves:

- (i) deleting a subword (s, s) ,
- (ii) replacing an alternating subword $(s, t, s, \dots)$ of length m_{st} by the alternating subword $(t, s, t, \dots)$ of the same length, called a **braid move**.

Moreover, two reduced words represent the same element of W if and only if they are related by braid moves.

Proof. See [Dav08, Section 3.4]. □

Proof of 3.1. (i) $\implies$ (ii) direction: by Lemma 2.9, (X, R) is a pre-reflection system. By Lemma 3.5, $X \setminus \mathcal{H}_r$ has at most two components, so it is enough to show $\mathcal{H}_r$ separates X . WLOG it suffices prove this for $\mathcal{H}_s \in S$, where $w \cdot \mathcal{H}_s = \mathcal{H}_{ws w^{-1}}$. By Lemma 3.6(i), since $n(s, s) = 1$, it follows that any path from e to s in X must cross $\mathcal{H}_s$ an odd number of times, in particular at least once. Hence e and s lie in separate components of $X \setminus \mathcal{H}_s$.

(ii) $\implies$ (iii) direction: we show that $\underline{s}$ is a reduced word iff the r_i in a reflection sequence for $\underline{s}$ are pairwise distinct. The result would then follow from Lemma 3.4.

$\implies$ direction follows from Lemma 3.4. For $\Leftarrow$ direction, assume that r_i are pairwise distinct. Let $w = s_1 \cdots s_k$ and

$$R(e, w) = \{r \in R \mid e, w \text{ lie in distinct components of } X \setminus \mathcal{H}_r\}.$$

For $r \in R(e, w)$, any path from e to w , and hence any word for w , must cross $\mathcal{H}_r$ at least once (see Figure 20). It follows that every $r \in R(e, w)$ appears in the reflection sequence of any word for w , so

$$l_S(w) \geq |R(e, w)|.$$

Since (X, R) is a reflection system, each $X \setminus \mathcal{H}_r$ has exactly two components, and crossing $\mathcal{H}_r$ moves between them. As the r_i are pairwise distinct, the path for $(s_1, \dots, s_k)$ crosses each $\mathcal{H}_{r_i}$ exactly once, so e and w lie in distinct components of $X \setminus \mathcal{H}_{r_i}$, i.e. $r_i \in R(e, w)$. Hence $|R(e, w)| \geq k$, so $l_S(w) \geq k$, and $(s_1, \dots, s_k)$ is reduced.

(iv) $\implies$ (i) suppose W is as in (iv). We want to show that (W, S) is a Coxeter system. Let m_{ij} be the order of $s_i s_j$ in W . Let (W', S') be the Coxeter system with matrix (m_{ij}) and generating set $S' = \{s'_i\}_{i \in I}$. Let $\varphi : W' \twoheadrightarrow W$ be a surjective homomorphism from the universal property of group presentation. It suffices to show that φ is injective, as it then $W' \cong W$, so (w, S) is a Coxeter system.

Let $w' \in \ker \varphi$ and $w' \neq e$. Then w be represented by a reduced word $(s'_1, \dots, s'_k)$ in W' . Since $\varphi(w) = e$, then $(s_1, \dots, s_k)$ is not reduced. Since W satisfies the exchange condition, by Tits' solution to the word problem (Theorem 3.7), $(s_1, \dots, s_k)$ can be shortened by braid moves and deleting (s, s) subwords. As these moves are also valid in W' , they shorten $(s'_1, \dots, s'_k)$ too. Hence $(s'_1, \dots, s'_k)$ not reduced. Contradiction. Therefore $\ker \varphi = e$. $\square$

Remark. How active is Coxeter groups research these days?

1k papers with Coxeter groups in title, of which about a hundred were in the last year. Artin groups, same presentation but without presentation, more than 1k papers.

Remark. The following is non-examinable.

Definition 3.8 (Space of Configurations). The space of configurations

$$\text{Conf}_n(\mathbb{C}) = \mathbb{C}_{\partial(z_1, \dots, z_n)}^n \setminus \{z_i \neq z_j \forall i \neq j\}.$$

See drawing. Since S_n acts on $\text{Conf}_n(\mathbb{C})$, we can quotient out

$$\text{Conf}_n(\mathbb{C})/W(A_n) = \text{UConf}_n(\mathbb{C}),$$

unordered configurations. Take π_1 ,

$$\pi_1(\text{Conf}_n(\mathbb{C})/W(A_n)) = A(A_n),$$

Artin group

$$A(A_n) = \langle \sigma_1, \dots, \sigma_n \mid \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \rangle.$$

Then $W(\Gamma) \rightsquigarrow M(\Gamma)/W_\Gamma$, then $\pi_1(M(\Gamma)/W_\Gamma) \cong A(\Gamma)$. Link to algebraic geometry.

For many Γ we can show $\pi_i(m(\Gamma)/W_\Gamma) = 0 \forall i \geq 2$, hence $m(\gamma)/W_\Gamma$ is a $K(A(\Gamma), 1)$.

Conjecture ($K(\pi, 1)$). For every Γ ,

$$m(\Gamma)/W_\Gamma \cong K(A_\Gamma, 1).$$

Remark. This implies solution for word problems to Artin groups and whether they are torsion free.

4 The Tits Representation

Aim. We want to prove that the Tits representation is faithful.

Theorem 4.1 (Tits). Let I be a finite indexing set, $S = \{s_i\}_{i \in I}$ and $M = \{m_{ij}\}_{i,j \in I}$ be a Coxeter matrix. Then there is a faithful representation

$$\rho : W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle \longrightarrow \mathrm{GL}_n(\mathbb{R})$$

where $n = |S| = |I|$, such that

- (i) $\forall i \in I, \sigma_i = \rho(s_i)$ is a linear involution with fixed point set a hyperplane,
- (ii) $\forall i, j \in I, \rho(s_i s_j)$ has order m_{ij} .

Remark. Point (ii) is not really a consequence of faithfulness, as we didn't know that $(s_i s_j)$ has order m_{ij} .

Remark. $\sigma_i \in \mathrm{GL}(\mathbb{R})$ won't usually be an orthonormal reflections.

Remark. This just tells you how much of a complex beast linear groups are.

4.1 Construction of the Tits Representation

Let (W, S) be as above, and take WLOG $I = \{1, \dots, n\}$, $V = \mathbb{R}^n$ with basis $\{e_1, \dots, e_n\}$.

Remark. Out of the blue.

Definition 4.2. We define a symmetric bilinear form B on V by

$$B(e_i, e_j) = \begin{cases} -\cos(\pi/m_{ij}) & \text{if } m_{ij} < \infty \\ -1 & \text{if } m_{ij} = \infty \end{cases}$$

Remark. Note that $B(e_i, e_i) = 1, B(e_i, e_j) \leq 0$ for $i \neq j$.

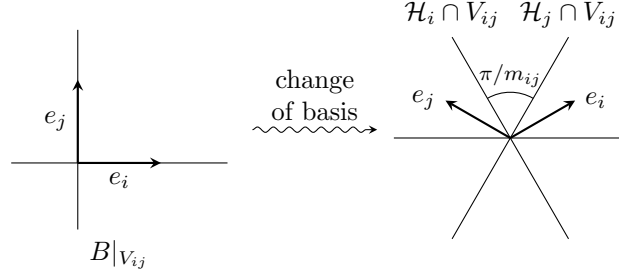
Remark. This will not be a metric in general.

Definition 4.3. We define

$$\begin{aligned} \sigma_i : V &\longrightarrow V \\ v &\longmapsto v - 2B(e_i, v)e_i. \end{aligned}$$

Proposition 4.4. We have

- (i) σ_i is a linear map
- (ii) $\sigma_i(e_i) = -e_i$
- (iii) Fixed points of σ_i are $\mathrm{Fix}(\sigma_i) = \{v \in V \mid B(e_i, v) = 0\} =: \mathcal{H}_i$ are hyperplanes.

Figure 21: Change of basis with respect to $B|_{V_{ij}}$

(iv) σ_i preserves the bilinear form: $B(\sigma_i(e_j), e_k) = B(e_j, e_k)$.

Proposition 4.5. $\sigma_i \sigma_j$ has order m_{ij} for all $i, j \in I$.

Proof. If $i = j$, this is trivial, hence assume $i \neq j$ and let $V_{ij} = \text{span}(e_i, e_j)$. We have $\sigma_i(V_{ij}) = V_{ij} = \sigma_j(V_{ij})$ so consider $\sigma_i \sigma_j|_{V_{ij}}$. If m_{ij} is finite, then the matrix representation of $B|_{V_{ij}}$ with respect to (e_i, e_j) is

$$\begin{pmatrix} 1 & -\cos(\pi/m_{ij}) \\ -\cos(\pi/m_{ij}) & 1 \end{pmatrix},$$

with $\det > 0, \text{rk} > 0$ so $B|_{V_{ij}}$ the definition. After a change of basis, get standard inner product in $\mathbb{R}^2$. Hence the angle between e_i, e_j is $-\cos(\pi/m_{ij}) = \cos(\pi - \pi/m_{ij})$, as in Figure 21. Thus $\sigma_i|_{V_{ij}}$ is an orthonormal reflection in $\mathcal{H}_i \cap V_{ij}$ and similarly for $\sigma_j|_{V_{ij}}$.

Upshot: $\sigma_i \sigma_j|_{V_{ij}}$ is a rotation by angle $2\pi/m_{ij}$, which gives order m_{ij} . Define

$$V_{ij}^\perp := \{w \in V \mid B(w, v) = 0 \forall v \in V_{ij}\},$$

and note that $V \cong V_{ij} \oplus V_{ij}^\perp$, and $\sigma_i \sigma_j|_{V_{ij}} = \text{id}$. Hence $\sigma_i \sigma_j$ has order m_{ij} on V , as required. This proves ρ is a representation.

Suppose now that $m_{ij} = \infty$, which gives matrix representation of $B|_{V_{ij}}$ with respect to (e_i, e_j) as $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, which is semi-definite but not definite. We calculate $\sigma_i \sigma_j(e_i) = \sigma_i(e_i + 2e_j) = e_i + 2(e_i + e_j)$. By iterating, we get $(\sigma_i \sigma_j)^k(e_i) = e_i + 2k(e_i + e_j)$, thus proving infinite order. $\square$

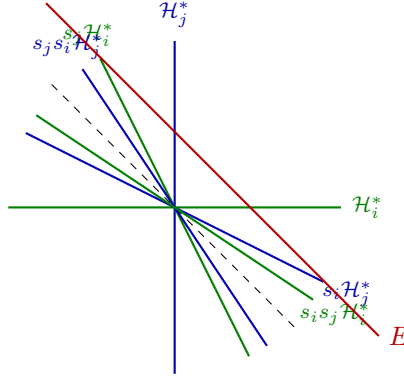
Corollary 4.6. The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho : W \longrightarrow \text{GL}_n(\mathbb{R})$.

Corollary 4.7. Let (W, S) be a Coxeter system. Then elements of S are pairwise distinct.

Proof. Suffices to show that $\sigma_i \neq \sigma_j$, for which we can use that $\sigma_i \sigma_j$ has order m_{ij} or just that they act differently on $\mathbb{R}^n$. $\square$

Corollary 4.8. $s_i s_j$ has order m_{ij} in W .

Proof. Immediate as $\sigma_i \sigma_j$ has order m_{ij} . $\square$

Figure 22: The case $m_{ij} = \infty$.

4.2 Geometry of $m_{ij} = \infty$

We have the matrix representation as $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, which has null space $N := \text{Null}(B|_{V_{ij}}) = \langle e_i + e_j \rangle$, $B|_{V_{ij}}$ induces a positive definite form on V_{ij}/N , which is one dimensional. Define $W_{ij} := \langle s_i, s_j \rangle \leq W$ and note that $W_{ij} \cong D_\infty$. But also $\mathcal{H}_i \cap V_{ij} = N = \mathcal{H}_j \cap V_{ij}$ is not a fruitful viewpoint.

Idea. We consider the dual vector space $V_{ij}^* = L(V_{ij}, \mathbb{R}) \ni \varphi$, with dual representation

$$(w \circ \varphi)(v) = \varphi(w^{-1}v),$$

where $w \in W_{ij}, \varphi \in V_{ij}^*, v \in V_{ij}$. This is faithful since it is the dual of a faithful representation.

Lecture 9

Let $\mathcal{H}_i^* = \{\varphi | \varphi(e_i) = 0\}$, and $Z = \{\varphi | \varphi(e_i + e_j) = 0\} = (V_{ij}/N)^*$.

Have $\rho^*(s_i) = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \rho^*(s_j) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ with respect to the dual basis, calculated with the formula in Definition 4.3 and then transposed.

Have $E = Z + 1$, $E \curvearrowright$ standard action of D_∞ . See Figure 22.

4.3 Faithfulness of the Tits Representation

Have dual representation $\rho^* : W \rightarrow \text{GL}(V^*)$. Have

$$(e^*(w)(\varphi))(v) = \varphi(e(w^{-1}))(v),$$

where $w \in W, \varphi \in V^*, v \in V$.

Aim. Want to prove ρ^* is faithful ($\iff \rho$ is faithful).

Define $\varphi_i \in V^*$ by

$$\varphi_i(v) = B(e_i, v),$$

Then $\sigma_i^* := \rho^*(s_i)$ is $\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$.

Define $\mathcal{H}_i^* := \{\varphi \in V^* | \varphi(e_i) = 0\}$ hyperplane and half-space $C_i := \{\varphi \in V^* | \varphi(e_i) > 0\}$.

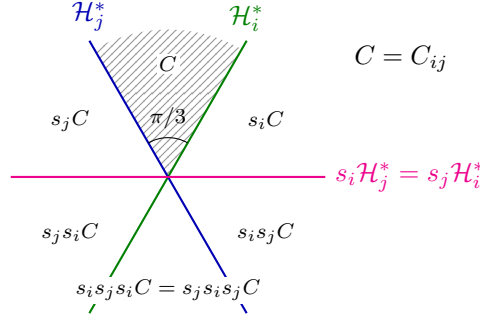


Figure 23: An example for m_{ij} finite.

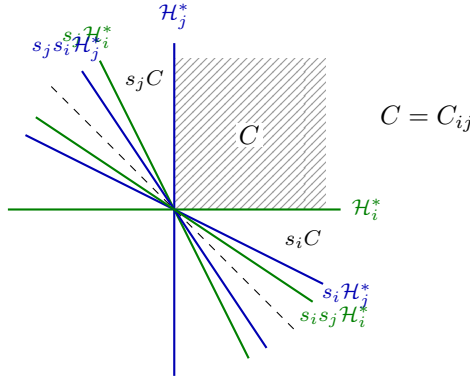


Figure 24: An updated version of Figure 22.

Definition 4.9 (Chamber of Representation). We define the **chamber of the representation** to be the closure $\overline{C}$ of $C := \bigcap_{i \in I} C_i$.

Also have $C_{ij} := C_i \cap C_j$.

Example (m_{ij} finite). Consider $\text{span}(\varphi_i, \varphi_j)$ identified with $\mathbb{E}^2$ with standard inner product. If $m_{ij} = 3$ get Figure 23 of interesting 2d slice (after passing to standard inner product). We have $V^* = \text{span}(\varphi_i, \varphi_j) \oplus (\mathcal{H}_i^* \cap \mathcal{H}_j^*)$, where s_i, s_j act on the first summand as in the figure and fix the second pointwise.

Example (m_{ij} infinite). Consider the 2-dimensional slice containing φ_i, φ_j . Again s_i, s_j fix $\mathcal{H}_i^* \cap \mathcal{H}_j^*$ pointwise, but now this meets $\text{span}(\varphi_i, \varphi_j)$, as $\varphi_i + \varphi_j$ vanishes on e_i and e_j . See Figure 24.

Definition 4.10. Let G be a group acting on a set Π . We say $\mathcal{C} \subset \Pi$ is **prefundamental** for G if

$$\forall g \in G, g\mathcal{C} \cap \mathcal{C} \neq \emptyset \implies g = 1 \in G.$$

Example. C_i is prefundamental for $W_i = \langle s_i \rangle$ acting on V^* .

Example. C_{ij} is prefundamental for W_{ij} .

Theorem 4.11 (Tits). $(W, S, \{C_i\})$ satisfies property P : for any $w \in W, i \in I$, either $wC \subset C_i$ or $wC \subseteq s_i C_i$. Moreover, in the second case, $l(s_i w) = l(w) - 1$ (word length with respect to S).

Idea. If there were only two generators, we already would satisfy P , as we already have $(W_{ij}, \{s_i, s_j\}, \{C_i, C_j\})$. So our strategy will be to define two properties: P_n : (P true for all w with $l(w) = n$) and Q_n : ($\forall w \in W$ with $l(w) = n, i \neq j, \exists u \in W_{ij}$ such that $wC \subset uC_{ij}$ and $l(u^{-1}w) = l(w) - l'(u)$), where l' is with respect to s_i, s_j .

Proof. P_0, Q_0 hold. Want to show that $(P_n \wedge Q_n) \implies P_{n+1}$ and $(P_{n+1} \wedge Q_n) \implies Q_{n+1}$.

Suppose $l(w) = n + 1, s_i \in S, w = s_j w'$ for some $j, l(w') = n$.

- (i) If $i = j$, apply P_n to w' . Must have $w'C \subset C_i \implies s_i w'C \subset s_i C_i$ and $l(s_i w) = nv$.
- (ii) If $i \neq j$, apply Q_n to w' . $\exists u \in W_{ij}$ such that $w'C \subset uC_{ij}, l(u^{-1}w') = l(w') - l'(u)$. Have that either $s_j uC_{ij} \subset C_i \implies wC \subset C_i$ or $s_j uC_{ij} \subset s_i C_i \implies wC \subseteq s_i C_i$. In second case, we're not done yet. What is the word length for the second case? Have

$$\begin{aligned} l(s_i w) &= l(s_i s_j w') \leq l(s_i s_j u) + l(u^{-1}w) \\ &\leq l'(s_i s_j u) + l(u^{-1}w) \\ &\leq l'(s_j u) - 1 + l(w') - l(u) \leq l(u) - 1, \end{aligned}$$

which must be an equality.

Assume now that $l(w) = n + 1, i \neq j$. If $wC \subset C_{ij}$, we're done as $u = 1$, hence assume not. Without loss of generality, we have $wC \not\subset C_i$. By $P_{n+1}, wC \subset s_i C_i$ and $l(s_i w) = l(w) - 1$. By $Q_n, \exists v \in W_{ij}$ such that $s_i wC \subset vC_{ij}$ and $l(s_i w) = l'(v) + l(v^{-1}s_i w)$. Then $wC \subset s_i vC_{ij}$ and

$$l(w) = 1 + l(s_i w) = 1 + l'(v) + l(v^{-1}s_i w) \geq l'(s_i v) + l((s_i)^{-1}w) \geq l(w),$$

which must both be equalities and Q_{n+1} holds with(out?) $u =_f v$. □

Corollary 4.12 (ES#2). C is prefundamental for W .

Corollary 4.13 (Tits Representation Faithful). ρ^* , and thus ρ , are faithful.

Lecture 10

Notation. From now on, we replace $C \mapsto \hat{C}, \bar{C} \mapsto C$.

Example 4.14 (D_{2n}). For n finite, then $V^* \leftrightarrow \mathbb{E}^2$ and the Tits cone is all of $\mathbb{E}^2$.

Example 4.15 (D_∞). $V^* = V_{ij}^*$ Tits cone is

$$\{\varphi \in V_{ij}^* \mid \varphi(e_i + e_j) > 0\} \cup \{0\},$$

where the point 0 is added due to taking closure, and the interior is the open half space bounded by Z and containing Z . See Figure 24.

5 Finite Coxeter groups

Aim. Coxeter-Dinkyn diagrams for classification.

Definition 5.1 (Reducible). A Coxeter system (W, S) is **reducible** if $S = S' \amalg S''$ such that

$$\forall s_i \in S', s_j \in S'' : m_{ij} = 2.$$

We say (W, S) is irreducible if it's not reducible.

Proposition 5.2 (Reducible Factor). If (W, S) is reducible, then

$$W \cong \langle S' \rangle \times \langle S'' \rangle.$$

Remark. The converse does not hold: W can be irreducible and still split as a product. For example see exercise on k odd, $D_{2(2k)} \cong D_{2k} \times C_2$.

Theorem 5.3. Let (W, S) be irreducible and $|S| = n$. The following are equivalent:

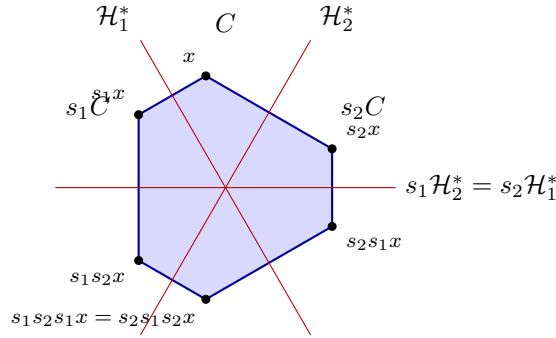
- (i) W is a geometric reflection group on $\mathbb{S}^{n-1}$ generated by $S = \{s_i\}_{i \in I}$, where s_i is the reflection across the codimension-one face F_i of a simplex in $\mathbb{S}^{n-1}$ such that F_i and F_j meet at angle π/m_{ij} .
- (ii) B is positive definite,
- (iii) W is finite.

Proof. See Section 6 of [Dav08], uses Thm 1.11. □

Remark. Similar theorems exist for Euclidean space, B positive semi-definite of corank 1, and Hyperbolic, with more complicated property (see Section 6 of [Dav08]).

If W finite $|S| = n$, then $V^*c \rightarrow \mathbb{E}^n$ and C is a closed Euclidean simplicial cone with boundary given by Hyperplanes.

Recall Corollary 4.12, which says $w \in W$, then if $w\mathring{C} \cap \mathring{C} \neq \emptyset$, then $w = e$. Hence if $x \in \mathring{C}$, then the orbit Wx has $|W|$ points.


 Figure 25: The Coxeter polytope for D_6 .

Definition 5.4 (Coxeter Polytope). Let (W, S) be finite. The **Coxeter polytope** for W is the convex hull of the W orbit on V^* of a point $x \in \overset{\circ}{C}$.

Remark. These are convex Euclidean polytopes, but are not in general regular.

Remark. This means that all *finite* Coxeter groups are geometric reflection groups.

Example 5.5 (D_6). See Figure 25. The 1-skeleton is isomorphic as non-metric graph to $\text{Cay}_S(W)$.

Aim. Coxeter classification of finite Coxeter groups: want to classify forms B associated to irreducible Coxeter systems by graphs by graphs.

Definition 5.6 (Coxeter-Dynkin Diagram). A **Coxeter-Dynkin diagram** Γ is a simply labelled graph with finite index set $V(\Gamma) = S = \{s_i\}_{i \in I}$ and edge labels

$$s_i \xrightarrow{m_{ij}} s_j$$

where $m_{ij} \geq 3$ or $m_{ij} = \infty$.

Lemma 5.7. There is a one-to-one correspondence between Coxeter system (W, S) and Coxeter-Dynkin diagrams Γ .

Proof. We will construct a bijection

$$\begin{aligned} \{(W, S) \text{ Coxeter System}\} &\longleftrightarrow \{\Gamma \text{ Coxeter diagram}\} \\ S &\longleftrightarrow V(\Gamma), \end{aligned}$$

in the obvious way. □

Notation. We omit edge labels if $m_{ij} = 3$ for the rest of the course, i.e.

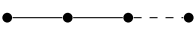
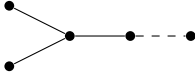
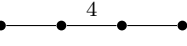
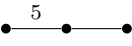
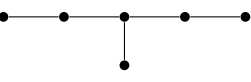
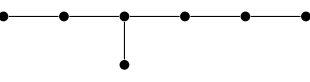
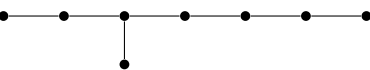
$$s_i \xrightarrow{3} s_j \longleftrightarrow s_1 \xrightarrow{\quad} s_j$$

Moreover, under the above bijection, we denote the image of Γ by $(W(\Gamma), V(\Gamma))$ or

$(W(\Gamma), S)$.

Remark. Many mathematicians use a different convention where no edge between s_i and s_j means $m_{ij} = \infty$ and not $m_{ij} = 2$, especially in Artin groups literature.

Theorem 5.8 (Coxeter '30s). If W is finite, then $(W, S) = (W(\Gamma), V(\Gamma))$ for Γ a disjoint union of a finite number of the following graphs, and conversely each of these graphs gives a finite Coxeter group:

- (i) $A_n, n \geq 1$: 
- (ii) $B_n, n \geq 2$: 
- (iii) $D_n, n \geq 4$: 
- (iv) $I_2(m), m \geq 5$: 
- (v) F_4 : 
- (vi) H_3 : 
- (vii) H_4 : 
- (viii) E_6 : 
- (ix) E_7 : 
- (x) E_8 : 

where for A_n, B_n, D_n the diagram has n vertices.

Example 5.9 (S_n). We have $S_n \cong W(A_{n-1})$.

Lecture 11

Remark. New correction to Lecture 9 typo.

Remark (B_n). B_n corresponds to ES#2 Coxeter Group of signed permutations.

Remark 5.10. If $\Gamma = \Gamma_1 \coprod \Gamma_2$ precisely when $(W(\Gamma), V(\Gamma))$ is reducible.

Example 5.11 (D_{2m}). We have D_{2m} as

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^M = e \rangle,$$

which gives

$$\Gamma = \begin{cases} W(A_2) & \text{if } m = 3 \\ W(B_2) & \text{if } m = 4 \\ W(I_2(m)) & \text{if } m \geq 5 \end{cases}$$

Example 5.12 (A_n). For $n \geq 2$, A_n , we have $W(\Gamma) \cong S_{n-1}$.

$$W(A_{n-1}) = \langle s_1, \dots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \rangle,$$

associated to the diagram:

$$s_1 \xrightarrow{(12)} s_2 \xrightarrow{(23)} s_3 \text{ -----}$$

Definition 5.13 (Full Subgraph). Let Γ be a Coxeter diagram and $S = V(\Gamma)$. Then Γ_T is the **full subgraph** of Γ spanned by $T \subseteq S$.

Remark. Then $(W(\Gamma_T), T)$ is a Coxeter system.

Definition 5.14 (Parabolic Subgroup). Let (W, S) be a Coxeter system and take $T \subseteq S$. The **parabolic subgroup** W_T of W is $W_T = \langle T \rangle$. If $T = \emptyset$, then $W_{\emptyset} = \{e\}$ is the trivial group.

Lemma 5.15. Let (W, s) be a Coxeter system and $W_T, (W(\Gamma_T), T)$ as defined above. Then $W(\Gamma_T) \cong W_T$.

Proof. Let $|S| = n$ and V an n -dimensional vector space with basis $\{e_s\}_{s \in S}$. Let $\rho : W \rightarrow \text{GL}(V)$ be the Tits representation with form B . Let G_T be the subgroup of $\text{GL}(V)$ which stabilises the subspace $V_T = \langle e_t \mid t \in T \rangle$. Now $(W(\Gamma_T), T)$ has its own Tits representation with form B_T and vector space $V = \langle e'_t \mid t \in T \rangle$. Then

$$\begin{aligned} \Phi : V' &\longrightarrow V \\ e'_t &\longmapsto e_t, \end{aligned}$$

is a vector space inclusion. By naturality of the Tits representation, $B|_T = B_T$, so we get a commutative diagram:

$$\begin{array}{ccc} W(\Gamma_T) & \xrightarrow{\rho_T} & \text{GL}(V') \\ \downarrow \text{UP of Group Pres} & & \uparrow V|_T \cong V' \\ W_T & \xrightarrow{\Phi|_T} & G_T \end{array}$$

The surjection is given by the UP of groups presentations and ρ being injective is given by Corollary 4.13, faithfulness of Tits representation. Thus the left arrow is an injection and $W(\Gamma_T) \cong W_T$. $\square$

Remark. There is also another way to prove this combinatorially.

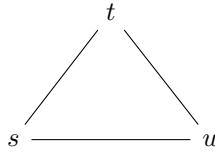
Definition 5.16 (Spherical Subgroup). If a parabolic subgroup is finite we call it a **spherical subgroup**.

Remark. Sometimes they are called special subgroups.

Corollary 5.17. All spherical subgroups can be obtained by observing Γ for (W, S) .

Proof. Apply Theorem 5.8 with Lemma 5.15. □

Example 5.18 (Triangle Group (3,3,3)). For $S = \{s, t, u\}$, we have a triangle



(W, S) has special subgroups $W_\emptyset, W_s, W_t, W_u$ of type $A_1, W_{\{s,t\}}, W_{\{s,u\}}, W_{\{t,u\}}$ of type A_2 .

Theorem 5.19. Let (W, S) be a Coxeter system. Then

- (i) For all $T \subseteq S$, (W_T, T) is also a Coxeter system.
- (ii) For all $T \subseteq S, w \in W_T$, $l_T(w) = l_S(w)$, i.e. $\text{Cay}_T(W)$ embeds isometrically into $\text{Cay}_S(W)$, and any reduced word for w in S $(s_1, \dots, s_k)$ satisfies $s_i \in T$.
- (iii) If $T, T' \subseteq S$, then $W_T \cap W_{T'} = W_{T \cap T'}$ and $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$.
- (iv) The bijection

$$\begin{aligned} T &\longleftrightarrow W_T \\ \{\text{subsets of } S\} &\longleftrightarrow \{\text{parabolic subgroups of } W\} \end{aligned}$$

preserves the partial order given by inclusion.

Lemma 5.20. Let (W, S) be Coxeter system, $w \in W$. Then $\exists S(w) \subseteq S$ such that given any reduced word $(s_1, \dots, s_k)$ for w , $S(w) = \{s_1, \dots, s_k\}$, i.e. $S(w)$ depends only on w .

Proof. Let w be of minimal length counter-example, i.e. $w = s_1 \cdots s_k = t_1 \cdots t_k$ with $s_i, t_i \in S$ and

$$\{s_1, \dots, s_k\} \neq \{t_1, \dots, t_k\}.$$

Then $w = s_1 v$ where $(s_2, \dots, s_k)$ is a reduced word for v . By the exchange condition, $l_S(s_1 w) < l(w)$ so $\exists i$ such that

$$w = s_1 t_1 \cdots \hat{t}_i \cdots t_k,$$

so v satisfies $S(v) \subseteq \{t_1, \dots, t_k\}$. Since $l_S(v) < l_S(w)$, it follows that

$$\{s_2, \dots, s_k\} = S(v) \subset \{t_1, \dots, t_k\}.$$

By the same argument on $w^{-1} = s_k \cdots s_1$, we get

$$\{s_{k-1}, \dots, s_1\} \subseteq \{t_1, \dots, t_2\},$$

and hence

$$\{s_1, \dots, s_k\} \subseteq \{t_1, \dots, t_k\}.$$

By symmetry, we can also show the reverse inclusion. Contradiction. $\square$

Lecture 12

Proof of 5.19. We have

- (i) This follows by Lemma 5.15.
- (ii) Let $w \in W_T$, then $S(w) \subset T$. Then by Lemma 5.20, if $(s_1, \dots, s_k)$ is a reduced word for $w \in W$, then each $s_i \in T$ so $l_S(w) = l_T(w)$.
- (iii) Clearly, $W_{T \cap T'} \subset W_T \cap W_{T'}$. To show $W_T \cap W_{T'} \subset W_{T \cap T'}$, we use Lemma 5.20 again, as if $w \in W_T \cap W_{T'}$, then $S(w) \subset T$ and $S(w) \subset T'$, so $S(w) \subset T \cap T'$, so $w \in nW_{T \cap T'}$. Showing that $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$ is an exercise.
- (iv) Let $T' \subset T$, we want to show $W_{T'} \subset W_T$. The by (iii) $W_{T \cap T'} = W_{T'} = W_T \cap W_{T'}$, so $W_{T'} \subseteq W_T$. Let $s \in T$ be such that $s \notin T'$. Then $S(w) = \{s\}$ so any reduced word representing s only involves s . Therefore $s \notin W_{T'}$. Hence $W_T' \subset W_T$.

$\square$

Notation. Given a Coxeter system (W, S) , we write

$$\mathcal{S} = \{T \subseteq S \mid W_T \text{ is spherical}\}.$$

Remark. This very much depends on the presentation (W, S) .

Remark. So $\mathcal{S}$ indexes the spherical subgroups, i.e. the finite parabolic subgroups W_T of W , and it is closed under taking subsets, since subgroups of finite groups are finite.

Question. Which topological spaces have Coxeter groups as fundamental groups?

6 The basic construction

Idea. Start with a group and family of subgroups, create a geometric realisation of the group with respect to that family of subgroups.

Idea. A geometric realisation is a geometric object with a nice action of the group.

Idea. The basic construction $\mathcal{U}(W, S)$ gives you a way to construct different geometric realisations.

Definition 6.1 ((Abstract) Simplicial Complex). An **abstract simplicial complex** is a (possibly infinite) set V , the **vertex set** and a collection X of finite subsets of V such that

- (i) $\forall v \in V, \{v\} \in X$,
- (ii) If $\Delta \in X, \Delta' \subseteq \Delta$, then $\Delta' \in X$.

An element Δ is called an **(abstract) simplex**. If $\Delta' \subsetneq \Delta$, we say that Δ' is a **face** of Δ . We define the **dimension** of Δ to be

$$\dim(\Delta) := |\Delta| - 1,$$

and call Δ a **k -simplex** if $\dim(\Delta) = k$, with the convention that every 0-simplex is a vertex.

The **k -skeleton** is

$$X^{(k)} = \bigcup_{\Delta \in X, \dim(\Delta) \leq k} \Delta.$$

We define

$$\dim(X) = \max\{\dim(\Delta) \mid \Delta \in X\},$$

and if $\dim(X) < \infty$, we say that X is **finite dimension**.

Definition 6.2 (Δ^n). The **standard n -simplex** Δ^n is the convex hull of the standard basis $e_1, \dots, e_{n+1} \in \mathbb{R}^{n+1}$.

$$\begin{aligned} \text{abstract simplicial complex} &\longrightarrow \text{simplicial cell complex} \\ n\text{-simplex } \Delta &\longmapsto \text{standard } n\text{-simplex } \Delta^n \\ \Delta' \text{ a face of } \Delta &\longmapsto \text{glue accordingly,} \end{aligned}$$

with appropriate inverse correspondence.

Definition 6.3 (Mirror Structure). Let (W, S) be a Coxeter system and X a connected Hausdorff topological space. A **mirror structure** on X over S is a family $(X_s)_{s \in S}$ of closed non-empty subsets of X . We call X a **mirrored space** over S and X_s the **s -mirror** of X .

Remark. In general, we can define mirror structures on arbitrary family of subgroups. See Section 5.1 in [Dav08].

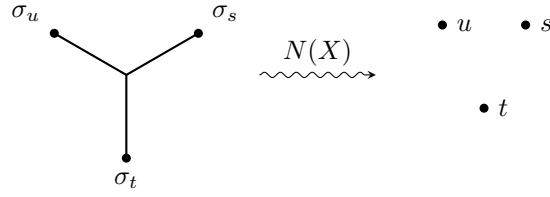


Figure 26: Nerve realisation of cone on $\{s, t, u\}$.

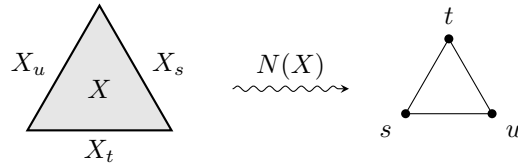


Figure 27: Nerve realisation of $X = \Delta^n$ for $n = 2$.

Idea. We obtain $\mathcal{U}(W, X)$ by gluing $|W|$ copies of X along mirrors.

Definition 6.4 (Nerve). Let (W, S) a Coxeter system, X is a mirrored space over S . The **nerve** $N(X)$ of X is an abstract simplicial complex with vertex set S and $T \subseteq S$ is a simplex if and only if $\bigcap_{t \in T} X_t \neq \emptyset$.

Example 6.5 (Cone). Consider

$$X = \text{Cone}\{\sigma_j | s \in S\},$$

i.e. a star graph with valence $|S|$. Let $X_s = \{\sigma_s\}$. Then for $S = \{s, t, u\}$, we have the situation depicted in Figure 26.

Example 6.6 ($X = \Delta^n$). For $X = \Delta^n$ where $|S| = n + 1$, then $|S|$ codimension-one faces labelled by S , we can take

$$\{\Delta_s | s \in S\},$$

and $X_S = \Delta_S$. See Figure 26.

Definition 6.7 (Convex Polytope). Let P^n convex polytope in $\mathbb{X}^n$, $n \geq 2$, $\{F_i\}_{i \in I}$, with $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$, or angle between them is π/m_{ij} , $m_{ij} \geq 2$. We set $m_{ii} = 1$. The Coxeter system (W, S) with matrix (m_{ij}) . Then we take $X = P^n$ and $X_{s_i} = F_i$ as a mirrored space.

Lecture 13

Example 6.8 (Chamber). Let $C \leq V^*$ be the chamber associated to the Tits representation H^* . See Figure 28. The dual HP fixed by $\sigma_i = \rho^*(s_i)$. So we take

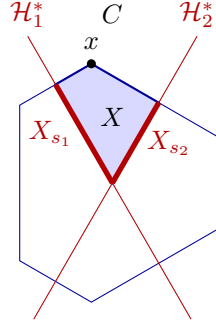


Figure 28: The chamber $X = C \cap (\text{Coxeter polytope})$ for D_6 , with its mirrors.

$$X = C, X_{s_i} = C \cap H_i^*.$$

Example 6.9. For W finite, $V^* \leftrightarrow \mathbb{E}^n$. So take

$$C = \{v \in \mathbb{R}^n \mid \langle v, e_i \rangle \geq 0 \forall i\},$$

and $x \in \overset{\circ}{C}$. The complex polytope is W_x , with $X = C \cap \text{Coxeter polytope}$ and $X_{s_i} = X \cap H_i$. See Figure 28 for D_6 .

Remark. From now on: (W, S) Coxeter system, X mirrored space over S ,

$$\exists x \in X. x \notin \bigcup_{s \in S} X_s.$$

Notation. We write, for every $x \in X$,

$$S(x) := \{s \in S \mid x \in X_s\}.$$

Warning. This is not the same as $S(w)$!

Example 6.10. Example 6.5, we have

$$S(x) = \begin{cases} \{S\} & \text{if } x = \sigma_s \\ \emptyset & \text{otherwise} \end{cases}$$

For Example 6.6, we have

$$S(x) = \begin{cases} \emptyset & \text{if } x \in \overset{\circ}{X} \\ T \subsetneq & \text{if } x \in \bigcap_{t \in T} \Delta_t \end{cases}$$

Definition 6.11 (Basic Construction). Consider W with the discrete topology and $W \times X$ with the product topology. The **basic construction** is the space

$$\mathcal{U}(W, X) = W \times X / \sim,$$

with the quotient topology, where

$$(w, x) \sim (w', x') \iff x = x' \wedge w^{-1}w' \in W_{S(x)},$$

Notation. We write $[w, x]$ for the equivalence class of (w, x) in $\mathcal{U}(W, X)$.

Example 6.12. If $x \in X_s$, then $s \in S(x)$, so $(w, x) \sim (ws, x) \forall w \in W$.

Remark. Thus $[w, x]$ contains at least (w, x) and (ws, x) .

Definition 6.13 (Chamber). For any $w \in W$, write wX for $\{w\} \times X$ in $\mathcal{U}(W, X)$. We call wX a **chamber** of $\mathcal{U}(W, X)$. The **fundamental chamber** is eX ,

Remark. wX and wsX are identified along X_s .

Lemma 6.14 (Cayley Basic). For X as in 6.5, up to subdivision, $\mathcal{U}(W, X)$ is $\text{Cay}_S(W)$.

Proof. Let $x \in X$. Then if $x \notin \{\sigma_s | s \in S\}$. We have $W_{S(x)} = W_\emptyset = \{e\}$, hence

$$(w, x) \sim (w', x) \iff w^{-1}w' \in \{e\} \iff w = w'.$$

Hence

$$[w, x] = \{(w, x)\}.$$

Otherwise, $x \in \{\sigma_s | s \in S\}$, $x = \sigma_s$, $s \in S$. Then $W_{S(x)} = W_{\{s\}} = \{e, s\}$.

We write

$$\begin{aligned} (w, x) \sim (w', x) &\iff w^{-1}w' \in \{e, s\} \\ &\iff w = w' \vee w' = ws, \end{aligned}$$

and thus $[w, x] = \{(w, x), (ws, x)\}$. Therefore in $\mathcal{U}(W, X)$, we glue wX and wsX along $X_s = \{\sigma_s\}$ and these are all the glueings. If we label the star of points of wX by w , then this gives the Cayley graph $\text{Cay}_S(W)$ with the edges subdivided by σ_s and mirror labels correspond to the edge labels in $\text{Cay}_S(W)$. $\square$

Example 6.15 (Triangle $(3, 3, 3)$). Take $\Gamma = \{s, t, u\}$ for the $(3, 3, 3)$ triangle group and X as in Example 6.5. We have situation as in Figure 29.

Definition 6.16 (Coxeter Complex). Let X be the mirrored space in Example 6.6, i.e. X is a simplex with codimension-one faces $\{\Delta_s | s \in S\}$. Then we call $\mathcal{U}(W, X)$ the **Coxeter complex**.

Example 6.17 (Triangle Group $(3, 3, 3)$). The Coxeter Complex for the triangle group $(3, 3, 3)$ is represented in Figure 30. If $x \in X_s \cap X_t$, then $W_{S(x)} = \langle s, t \rangle \cong D_6$, hence for all $w \in W$, wX is glued to $wX, wsX, wtX, wstX, wtsX$ and $wstsX$ at the point $x \in X_s \cap X_t$. Note that $\mathcal{U}(W, X)$ is a tessellation of $\mathbb{E}^2$ by triangles.

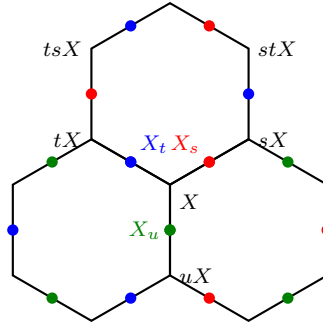


Figure 29: $\mathcal{U}(W, X)$ for the triangle group $(3,3,3)$.

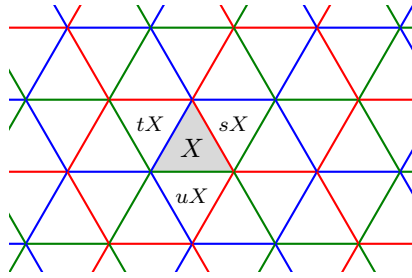


Figure 30: The Coxeter Complex for the triangle group $(3,3,3)$.

Remark. People that do representation theory and study building really care about Coxeter complexes!

Remark. If W is an irreducible finite Coxeter group, then the Coxeter complex can be identified with the tessellation of the sphere by spherical simplices induced by W . If W is *affine*, then there exists an affine subspace $E \subset V^*$ given by slicing across the interior of the Tits' cone. Then $W \curvearrowright E$ by isometries and Coxeter complex corresponds to a tessellation of $\mathbb{E}^n$ by intersection E with the interior of the Tits' cone (see Figure 24).

Lemma 6.18. $\mathcal{U}(W, X)$ is a connected topological space.

Proof. We note that since $\mathcal{U}(W, X)$ has quotient topology, $A \subseteq \mathcal{U}(W, X)$ is open (or closed) if and only if $A \cap w \cdot X$ is open (or closed) for each $w \in W$.

Let $\emptyset \neq A \subset \mathcal{U}(W, X)$ and assume that A is open and closed. Thus $A \cap w \cdot X$ open and closed, but X is connected, hence $A \cap w \cdot X$ is $\emptyset$ or X . This gives

$$A = \bigcup_{v \in V} v \cdot X, \emptyset \neq V \leq W.$$

For $v \in V$, $s \in S, \exists x \in X_s$, where $X_s \neq \emptyset$, and $[vs, x] = [v, x]$. Thus any open neighbourhood at $[v, x]$ intersects $vs \cdot X$. Therefore $vs \in V$. Since s was arbitrary, we have $VS \subseteq V$, but $W = \langle S \rangle$, so $V = W$ and $A = \mathcal{U}(W, X)$. $\square$

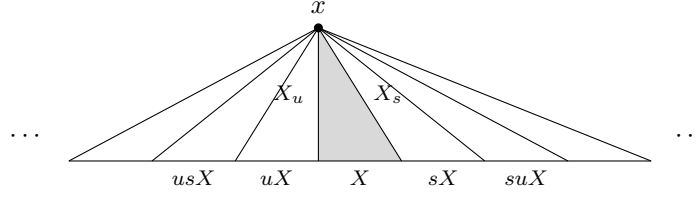


Figure 31: If $m_{su} = \infty$, infinitely many chambers of the Coxeter complex are glued at a point $x \in X_s \cap X_u$.

Definition 6.19 (Locally Finite). $\mathcal{U}(W, X)$ is **locally finite** if for all $[w, x] \in \mathcal{U}(W, X)$, there is an open neighbourhood which meets only finitely many chambers.

Example 6.20 (Cayley Graph). $\text{Cay}_W(S)$ as $\mathcal{U}(W, S)$ is locally finite, see Lemma 6.14. See Figure 29.

Example 6.21 (Coxeter Complex). The Coxeter complex is not always locally finite. For the triangle group $(3, 3, 3)$ in Example 6.17 it is. However, for D_∞ it isn't.

But the Coxeter Complex for Γ has $x \in X_s \cap X_u$ with $m_{su} = \infty$, see Figure 31. Have $W_{S(x)} \cong D_\infty$, so it has infinitely many chambers glued to eX at (e, x) .

Lemma 6.22. The following are equivalent:

- (i) $\mathcal{U}(W, X)$ is locally finite,
- (ii) $\forall x \in X$, $W_{S(x)}$ is finite,
- (iii) $\forall T \subseteq S$ such that W_T is infinite, $\bigcap_{t \in T} X_t = \emptyset$.

Proof. (ii) $\iff$ (iii) direction is immediate.

(i) $\implies$ (iii) direction: if (iii) not true, then $\exists x$ with $S_{(x)}$ infinite. Then an infinite number of chambers are identified at $[e, x]$, and thus (i) is not true.

(ii) $\implies$ (i) direction: for each $[w, x] \in \mathcal{U}(W, X)$ we can find an open neighbourhood U which only intersects chambers $w'X$ with $w^{-1}w' \in W_{S(x)}$. Then by (ii), $W_{S(x)}$ is finite, and thus U intersects with a finite number of chambers, and thus (i) holds. $\square$

Definition 6.23 ($W \curvearrowright \mathcal{U}(W, X)$). The **action** of W on $\mathcal{U}(W, X)$ by homeomorphisms is given by acting to the left of $W \times X$, i.e.

$$w' \cdot (w, x) = (w'w, x).$$

Since this preserves the equivalence relation, we get an action on $\mathcal{U}(W, X)$.

Lemma 6.24. X is a strict fundamental domain for $W \curvearrowright \mathcal{U}(W, X)$.

Proof. ES#4. $\square$

Corollary 6.25. $\mathcal{U}(W, X)/W \cong X$.

Proof. Trivial. $\square$

Remark. Moreover, we have $w' \cdot (w \cdot X) = w'w \cdot X$ gives a transitive, free action of W on the set of chambers of $\mathcal{U}(W, X)$.

Lemma 6.26. $\text{Stab}_W([w, x]) = \{w' \in W \mid w^{-1}ww' \in W_{S(x)}\} = wW_{S(x)}w^{-1}$.

Proof. Trivial. $\square$

Lemma 6.27. $\mathcal{U}(W, X)$ is Hausdorff.

Proof. Let $y = [w, x] \in \mathcal{U}(W, X)$ and $W_y = \text{Stab}_W([w, x])$. Then for $x \in U_x \subset X$ an open neighbourhood, then $V_y = W_y(U_x - \bigcup_{x \notin X_s} X_s)$ is open in $\mathcal{U}(W, X)$.

If $y' = [w', x']$ such that $y \neq y'$, then U_x and $U_{x'}$ can be chosen small enough such that $V_y \cap V_{y'} = \emptyset$. $\square$

Definition 6.28 (Properly Discontinuous). G is a discrete group Y and Y is a Hausdorff space. An action by homeomorphisms $G \curvearrowright Y$ is **properly discontinuous** if

- (i) Y/G is Hausdorff,
- (ii) $\forall y \in Y. G_y = \text{Stab}_G(y)$ is finite,
- (iii) $\forall y \in Y \exists y \in U_y \subseteq O Y (G_y \cdot U_y = U_y \wedge \forall g \in G \setminus G_y. gU_y \cap U_y = \emptyset)$,

Lemma 6.29. $W \curvearrowright \mathcal{U}(W, X)$ is properly discontinuous iff for all $x \in X, W_{S(x)}$ is finite.

Proof. $\Leftarrow$ direction: conditions (i) and (ii) follow by Lemma 6.27 and Lemma 6.26.

Without loss of generality, we show (iii) for $[e, x]$. Then $V_y = W_{S(x)}(X - \bigcup_{x \notin X_s} X_s)$ from Lemma 6.27 satisfies $W_y \cdot V_y = V_y$ and $w \cdot V_y \cap V_y = \emptyset \forall w \in W \setminus W_{S(x)}$. See Figure 31 again. $\square$

Lecture 15

7 The Davis complex

Idea. Simplicial complex built out of a Coxeter group.

Remark. Recall

$$\mathcal{S}(W, S) = \{T \subseteq S \mid W_T \text{ is spherical (finite)}\} \ni \emptyset,$$

$$W_\emptyset = \{\text{id}\}.$$

Remark. In this section, abstract simplicial complexes do not have a $\emptyset$ simplex.

Definition 7.1 (Nerve). The **nerve** of (W, S) , denoted $L(W, S)$ is an abstract simplicial complex with vertex set S and simplex set $\mathcal{S} \setminus \emptyset$.

Example 7.2. See three drawings.

Example 7.3. Let $S = S_1 \coprod S_2$ with $m_{ij} = \infty \forall s_i \in S_1, s_j \in S_2$. Let $W_i = \langle S_i \rangle, i = 1, 2$, then

$$L(W, S) = L(W_1, S_1) \coprod L(W_2, S_2).$$

Definition 7.4 (Barycentric Subdivision). Let X be an abstract simplicial complex. Then its **barycentric subdivision** is an abstract simplicial complex X' with vertex set X and simplex set

$$X' = \{\{\Delta_0, \dots, \Delta_p\} | p \in \mathbb{N}, \Delta_i \in \Delta_{i+1} \forall 0 \leq i \leq p-1\}.$$

Example 7.5. See drawing.

Definition 7.6 (Chamber). The **chamber** K of (W, S) is the cone on the barycentric subdivision on L' of the nerve $L = L(W, S)$.

Notation. Let $K_s \subset K$ be the star in L' of the vertex s , i.e.

$$K_s = \bigcup_{\sigma_s \in L', s \in \sigma_s} \sigma_s.$$

We label the cone point " $\emptyset$ ".

Example 7.7. Improvement on first examples of lecture. See drawing.

Remark. K is connected and Hausdorff, so $(K_s)_{s \in S}$ is a mirror structure on K .

Remark. K is the simplicial complex $\text{Flag}(\mathcal{S})$, an $\mathcal{S}$ -poset with inclusion.

Remark. $\emptyset$ is contained in no mirror.

Warning. In all our examples K is one-dimensional. This is not always the case.

Lemma 7.8. A mirrored space X for (W, S) satisfies $W_{S(x)}$ finite for all $\forall x \in X$ iff $N(X) \subseteq L(W, S)$.

Proof. $\implies$ direction: let $\{t_1, \dots, t_k\}$ simplex in $N(X)$. Then $\exists x \in \bigcap_{1 \leq i \leq k} X_{t_i}$, and thus $W_{\{t_1, \dots, t_k\}}$ is finite, hence $\{t_1, \dots, t_k\} \in \mathcal{S}$, thus $\{t_1, \dots, t_k\}$ simplex in $L(W, S)$.

$\impliedby$ direction: let $x \in \bigcap_{t \in T} X_t$. Then T is a simplex in $N(X)$. Hence T is a simplex in $L(W, S)$, thus $T \in \mathcal{S}$, and hence $W_T = W_{S(x)}$ is finite. $\square$

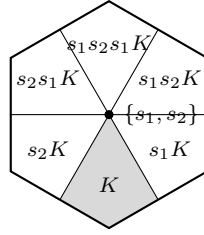


Figure 32: The Davis complex Σ for D_6 , with the chamber K shaded.

Corollary 7.9. K satisfies $N(K) = L(W, S)$, and thus for all $x \in K$, $W_{S(x)}$ is finite.

Proof. See ES#3. □

Definition 7.10 (Davis Complex). The **Davis complex**

$$\Sigma(W, S) = \mathcal{U}(W, K).$$

Corollary 7.11. $\Sigma(W, S)$ is connected, Hausdorff, locally finite. Moreover, W acts properly discontinuously on Σ with quotient K , and all point stabilisers are conjugates of spherical subgroups of W .

Remark. To compute the homology of a group, we take $\underline{EG}$ a contractible space under which the group acts freely, and quotient out by group action. See [Boy20].

Example 7.12. More drawings, D_6, D_∞ .

Example 7.13 (Triangle $(3, 3, 3)$). If W is the $(3, 3, 3)$ triangle group, then $\Sigma(W, S)$ is the barycentric subdivision of the tiling of $\mathbb{E}^2$ by triangles.

Remark. If W is a Euclidean or hyperbolic geometric reflection group, then $\Sigma(W, S)$ is the barycentric subdivision of the corresponding tessellation of $\mathbb{E}^n$ or $\mathbb{H}^n$ by P .

Moreover, if W is spherical, $\Sigma(W, S)$ be identified with the barycentric subdivision of the associated Coxeter polytope.

Remark. The Davis complex may be realised as a CW complex, where the cells are given by the polytopes given by the finite subgroups.

Lecture 16

Remark. Recall: $L(W, S) \rightarrow L'$ barycentric subdivision $\rightarrow K$ cone on L' , cone point $\emptyset$, $K_S = \text{Star}(\{s\})$ in $L' \subset K$. This gets us $\Sigma(W, S) = \mathcal{U}(W, K)$.

See Figure 32 for D_6 .

Theorem 7.14. The Davis complex $\Sigma(W, S)$ is contractible.

Example 7.15. For $W = D_6$, $\Sigma(W, S)$ is the hexagon of Figure 32, which is contractible.

Definition 7.16. For $w \in W$, we define

$$\begin{aligned} \text{In}(w) &:= \{s \in S \mid l_S(ws) < l_S(w)\}, \\ \text{Out}(w) &:= \{s \in S \mid l_S(ws) > l_S(w)\}. \end{aligned}$$

Example 7.17. Take $w = sts$ in $s \rightarrow t$ (undirected). Then $\text{In}(W) = \{s, t\}$. If $v = t$, $\text{In}(v) = \{t\}, \text{Out}(v) = \{s\}$.

Example 7.18. See drawing for $\{s, u, t\}$.

Remark. We have $l_S(ws) = l_S(w) \pm 1$, thus $S = \text{In}(w) \amalg \text{Out}(w)$.

Remark. If $l_S(ws) < l_S(w)$, then if $(s_1, \dots, s_k)$ is a reduced word for w , by (E) on $(ws)^{-1}$, we get $w^{-1} = ss_k \cdots \widehat{s_i} \cdots s_1$ for some $1 \leq i \leq k$. and thus $w = s_1 \cdots \widehat{s_i} \cdots s_k s$. So

$$\text{In}(w) = \{s \in S \mid \exists \text{ red. word for } w \text{ ending in } s\}.$$

Lemma 7.19. Let (W, S) be a Coxeter system. Suppose $\exists w_0 \in W$ such that for all $s \in S$, $l_S(sw_0) < l_S(w_0)$. Then W is finite.

Proof. By contradiction, on ES#3. □

Lemma 7.20. For $T \subset S$, there is a unique element ω of the minimal length in the coset wW_T , such that all elements $w' \in wW_T$ can be written in the form $w' = wa$ for some $a \in W_T$ with $l_S(w') = l_S(w) + l_S(a)$.

Proof. See ES#3. □

Proposition 7.21. For all $w \in W$, $\text{In}(w) \in \mathcal{S}$, i.e. $W_{\text{In}(w)}$ is finite.

Proof. Consider the coset $wW_{\text{In}(w)}$, and let u be the unique element of minimal length. By Lemma 7.20, $w \in wW_{\text{In}(w)}$, so $w = ua$ for some $a \in W_{\text{In}(w)}$, $l_S(w) = l_S(u) + l_S(a)$.

Moreover, $\forall s \in \text{In}(w), l_S(ws) < l_S(w)$, and $\forall s \in \text{In}(w), as \in W_{\text{In}(w)}$, so $ws = uas$ satisfies $l_S(ws) = l_S(u) + l_S(as)$. Hence

$$\forall s \in \text{In}(w), l_S(as) < l_S(a).$$

We now apply Lemma 7.19 to a^{-1} to get $\text{In}(w) \in \mathcal{S}$. □

Lemma 7.22. K is contractible and if $T \in \mathcal{S}$, then $K^T = \bigcup_{t \in T} K_t$ is contractible.

Proof. K is contractible since it is a cone, so we can contract to the cone point $\emptyset$.

For $\emptyset \neq T \in \mathcal{S}$, T spans a simplex in L , σ_T . Let σ'_T be the barycentric subdivision of σ_T in L' . Then σ'_T is contractible, so it's enough to construct $r : K^T \rightarrow \sigma'_T$, a deformation retract.

A vertex x in K^T corresponds to $T' \in \mathcal{S}$ such that $t \in T'$ for some $t \in T$, and thus $T \cap T' = \emptyset$. Hence we can x to the vertex of σ'_T corresponds to $T \cap T'$.

We extend this to simplices by mapping a simplex σ_V with vertices $\{v_0, \dots, v_k\}$ to simplex $\{T \cap V_0, \dots, T \cap V_k\}$. Note that $\sigma_V \in K^T \iff T \ni t \in V_i \forall 0 \leq i \leq k$. $\square$

Proof of 7.14. We list the elements of W as $w_1, w_2, \dots$ such that $l_S(w_n) \subseteq l_S(w_{n+1}) \forall n \geq 1$. If W finite repeat the last element to get an infinite list. Let $U_n = \{w_0, \dots, w_n\} \subseteq W$, so $W = \bigcup_{i \in \mathbb{N}} U_n$.

Let

$$P_N = \bigcup_{w \in U_n} wK = \bigcup_{i \in \mathbb{N}} w_i K = \Sigma(W, S),$$

so $P_i \subseteq P_{i+1}$ and $\Sigma(W, S) = \bigcup_{i \in \mathbb{N}} P_i$.

Clearly, $P_1 = w_1 K 0 = eK = K$ is contractible. For $P_n = P_{n-1} \cup w_n K$, we're gluing along mirrors. To decide which mirror to glue along, we look at

$$\{K_s | l_S(w_n s) < l(w_n)\} = \{K_s | s \in \text{In}(w_n)\},$$

so we glue along $K^{\text{In}(w_n)}$. By Proposition 7.21, $\text{In}(w_n) \in \mathcal{S}$, so by Lemma 7.22,

We have P_1 contractible and $P_{n-1} \rightsquigarrow P_n$ we glue on $w_n K$, contractible, along $K^{\text{In}(w)}$ contractible. At each stage P_n is contractible, so $\Sigma(W, S)$ is contractible. $\square$

Example 7.23 (D_6). We list the elements e, s, t, st, ts, sts . See drawing.

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