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MATHEMATICS TRIPOS

Part III

Infinite Groups

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0 Introduction

The goal of this course is to explore some of the universe of finitely generated infinite groups. By contrast with the world of finite groups, where powerful classification theorems are available, infinite groups can exhibit extremely exotic and wild behaviour, so we will be guided by examples.

After some generalities on definition of groups by generators and relations, we look at some particular classes of groups. Certainly the easiest class of finitely generated infinite groups to understand is the abelian groups, and we start by considering groups which can be constructed from abelian “building blocks”: the nilpotent and soluble groups. Next, by considering some properties that abelian and finite groups have in common, we arrive at the definition of an amenable group (a much larger class). Our discussion of amenability leads naturally to a beautiful application: the famous Banach-Tarski paradox.

We next turn to the concept of the “growth” of a finitely generated group. We investigate the growth of soluble groups, and observe that a soluble group grows either exponentially or polynomially. This begs the question: are any other types of growth possible? Answering this question, via a famous construction of Grigorchuk, is a highlight of the course, as it will give us the tools to answer several other famous questions about infinite groups, including problems posed by Burnside and von Neumann.

In the last chapter we investigate groups which can be “approximated” by finite or amenable groups, in various senses. We construct a metric space, the “space of marked groups” which (roughly) has as its points finitely generated groups, and the distance between two groups measures how well they approximate one another. Finally, we reach a class of approximable groups, the “sofic” groups, which is so large that it is not known whether every group lies inside it. We complete the course with a discussion of this famous unsolved problem, and some speculations as to where a non-sofic group might be found.

Reference is [Rob96a, Rob96b, Rob96c].

Lecture 1

1 Motivation: Three Unsolved Problems

1.1 Kaplansky’s Direct Finiteness Conjecture

Definition 1.1 (Direct finiteness). Let R be a ring with unity (not necessarily commutative). We say R is **directly finite** if $\forall a, b \in R$, if $ab = 1$ then $ba = 1$.

Example. Any commutative ring is directly finite.

Example (Non). Let V be the vector space of real sequences and take R to be the ring of linear maps on V . Define

$$\begin{aligned}\varphi, \lambda : V &\longrightarrow V \\ \varphi : (x_1, x_2, x_3, \dots) &\longmapsto (0, x_1, x_2, \dots) \\ \lambda : (x_1, x_2, x_3, \dots) &\longmapsto (x_2, x_3, x_4, \dots),\end{aligned}$$

and observe that $\lambda \circ \varphi = \text{id}_V$ but $\varphi \circ \lambda \neq \text{id}_V$.

Definition 1.2 (Group algebra). Let G be a group and k a field. The **group algebra** $k[G]$ is defined as

$$k[G] = \left\{ \sum_{g \in G} \lambda_g \cdot g \right\},$$

of finite formal k -linear combination of elements of G .

Proposition 1.3. $k[G]$ is a k -vector space with basis G . Moreover, $k[G]$ can be equipped with a ring multiplication by taking

$$(\lambda \cdot g) * (\mu \cdot g) = (\lambda\mu) \cdot (gh),$$

and extending linearly. The identity is $1 \cdot e$.

Definition 1.4 (Direct finiteness). A group G is directly finite if for all fields k , $k[G]$ is directly finite (w.r.t. the previous ring structure).

Exercise. The following hold

- (i) G abelian $\implies k[G]$ commutative,
- (ii) G finite $\implies G$ d.f. (on ES#1, need vector space structure)

Conjecture (Kaplansky). Every group is directly finite.

Theorem 1.5 (Kaplansky). For every G and k field of zero characteristic, $k[G]$ is directly finite.

Remark. Not known about arbitrary fields.

State of the art:

Theorem 1.6 (Elek+Szabo [ES03]). Every sofic group is directly finite.

Remark. The adjective "sofic" comes from the Hebrew word for finite. These groups represent a generalisation of finite groups.

1.2 Surjunctivity Conjecture

Remark. This subsection is not examinable.

Remark. In this course, all the group actions will be left actions.

Definition 1.7 (Surjunctive Topological Dynamical System). A **topological dynamical system** over G is a topological space X equipped with a G -action by homeomorphisms.

A **subsystem** of X is a closed G -invariant subspace of X .

An isomorphism of topological dynamical systems X, Y over G , is a homeomor-

phism $f : X \xrightarrow{\sim} Y$ such that for all $x \in X$ we have

$$g \cdot \varphi(x) = \varphi(g \cdot x).$$

A topological dynamical system is **surjunctive** if it is not isomorphic to any of its proper subsystems.

Definition 1.8 (Shift system). Let A be a finite discrete space, with $|A| \geq 2$. A **shift system** is an action $G \curvearrowright A^G$ (which is compact by Tychonoff), given by

$$g \cdot (a_h)_{h \in G} = (a_{gh})_{h \in G}.$$

We say G is **surjunctive** if every A , A^G is surjunctive.

Conjecture (Surjunctivity). Every group G is surjunctive.

Proposition 1.9. Any finite group is surjunctive.

Theorem 1.10 (Gromov 99). Every sofic group is surjunctive.

Remark. This theorem is what sofic groups were first defined for, albeit with different terminology.

1.3 Sofic Groups

The two previous conjectures require the existence of non-sofic groups to find counterexamples. However, we don't even know if any exist:

Conjecture. Every group G is sofic.

Remark. There is some belief that specific groups may be non-sofic. We may discuss examples at end of course.

To understand sofic groups, we should first go back to models of finite groups. Well-known theorem by Cayley:

Theorem 1.11 (Cayley 54). Let G be finite, then the symmetric representation

$$\begin{aligned} \varphi : G &\longrightarrow \text{Symm}(n) \\ \varphi(g) \cdot h &\longmapsto g \cdot h, \end{aligned}$$

is injective. Moreover, $\forall e \neq g \in G$, $\varphi(g)$ has no fixed points.

Corollary 1.12. We may view every finite group as subgroup of symmetric group that has no fixed points.

In order to perturb this notion of finiteness, we need a notion of distance.

Definition 1.13 (Hamming Distance). Let $\sigma, \tau \in \text{Symm}(n)$, the **hamming distance** from σ to τ is

$$d_n(\sigma, \tau) = 1 - \frac{1}{n} \# \{1 \leq i \leq n : \sigma(i) = \tau(i)\},$$

i.e. the fraction of fixed points of $\sigma\tau^{-1}$.

Remark. Hamming distance gives a metric.

Theorem 1.14 (Cayley Revisited). A group G is finite if and only if there exists some $n \in \mathbb{N}$ and $\varphi : G \rightarrow \text{Symm}(n)$ such that

- $\forall g, h \in G : d_n(\varphi(gh), \varphi(g)\varphi(h)) = 0$.
- $\forall e \neq g \in G : d_n(\text{id}_n, \varphi(g)) = 1$.

Definition 1.15 (Sofic). A group G is **sofic** if it (almost) respects the two previous conditions, i.e. if for all finite subsets $A \subset G$ and $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ and function $\varphi : A \hookrightarrow \text{Symm}(n)$ such that

- $\forall g, h \in A : gh \in A \implies d_n(\varphi(gh), \varphi(g)\varphi(h)) < \varepsilon$.
- $\forall e \neq g \in A : d_n(\text{id}_n, \varphi(g)) > 1 - \varepsilon$.

The φ we defined is called an (A, ε) -**representation**.

Proposition 1.16 (Finite Sofic). Every finite group is sofic.

Proof. Trivial. □

Want to prove that

Theorem 1.17 (Abelian Sofic). Every abelian group is sofic.

We first prove it for $\mathbb{Z}$:

Lemma 1.18 ($\mathbb{Z}$ sofic). $\mathbb{Z}$ is sofic.

Proof. Fix some finite $A \subset \mathbb{Z}$ and $\varepsilon > 0$, we want to find an (A, ε) -representation.

We can find some $m \in \mathbb{N}$ such that $A \subseteq \mathbb{Z} \cap [-m, m]$. Let $n \geq 2m + 1$, $\sigma = (1, 2, \dots, n) \in \text{Symm}(n)$.

Define

$$\begin{aligned} \varphi : \mathbb{Z} &\longrightarrow \text{Symm}(n) \\ k &\longmapsto \sigma^k, \end{aligned}$$

which is a homomorphism.

We may view this as the homomorphism that is given by taking the quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/(n)$ and then applying Cayley's theorem to $\mathbb{Z}/(n)$. Thus our σ would be the image of $[1] \in \mathbb{Z}/(n)$.

Have defined φ , we have that $\varphi|_A : A \rightarrow \text{Symm}(n)$ is an $(A, 0)$ -representation. Our ε plays no roles as there are no approximations: no power of σ has any fixed points until we get to powers that are at least n in absolute value. □

Want to prove:

Proposition 1.19. Every finite generated abelian group is sofic.

First want to reduce to finitely generated abelian groups, which is easy since

Lemma 1.20. G is sofic $\iff$ every finitely generated subgroup of G is sofic.

Proof. $\Leftarrow$. $A \subseteq G$ finite, $\varepsilon > 0$, $A \subseteq \langle A \rangle$ is finitely generated, so by assumption there is an (A, ε) -representation.

$\Rightarrow$. $H \leq G$ finitely generated, $A \subseteq H$ finite, then $A \subseteq G$ so there is an (A, ε) -representation. $\square$

Exercise (ES#1.2). Let X_1, X_2 be finite sets. Show that there is an injective homomorphism, given by

$$\begin{aligned} \Phi : \text{Symm}(X_1) \times \text{Symm}(X_2) &\longrightarrow \text{Symm}(X_1 \times X_2) \\ (\sigma_1, \sigma_2) \cdot (x_1, x_2) &\longmapsto (\sigma_1 \cdot x_1, \sigma_2 \cdot x_2) \end{aligned}$$

Deduce that if G and H are sofic groups, then so is $G \times H$.

Finally, proof of theorem:

Proof. G is abelian, can reduce to finitely generated subgroups, so take G to be finite generated. We can apply classification of finitely generated abelian groups to get $G \cong \mathbb{Z}^n \times T$, T finite. But $\mathbb{Z}$ is sofic and T is also sofic, and sofic groups are closed under direct products. $\square$

Lecture 2

2 Free groups

The following fact should be familiar:

Proposition 2.1. A free group of $\text{rk}(G)$ at least 2 has trivial centre.

We're now going to refresh some notions in relating to free groups. Let X be a set in what follows:

Definition 2.2 (Formal inverses). The **formal inverses** of X , denoted

$$X^{-1} = \{x^{-1} | x \in X\},$$

is another set bijecting with X .

Definition 2.3 (Word). A **word** in X is a finite sequence of symbols $\{y_1, y_2, \dots, y_n\}_n$, $y_i \in X \cup X^{-1}$. We call the case of $n = 0$, the **empty word**, denoted $\emptyset$ and denote $W(X)$ the set of words on X .

Definition 2.4 (Concatenation). We define **concatenation** of words to be

$$\begin{aligned} c : W(X) \times W(X) &\longrightarrow W(X) \\ (y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_l) &\longmapsto y_1 \cdots y_n z_1 \cdots z_l. \end{aligned}$$

Proposition 2.5 (Monoid). $W(X)$ together with concatenation forms a monoid with identity $\emptyset$.

Proof. Trivial. □

Definition 2.6 (Elementary contraction/expansion). For $w, v \in W(X)$, we say w **contracts to** v ($w \searrow v$), or v **expands to** ($v \nearrow w$), if there is some $x \in X, u_1, u_2 \in W(X)$ such that $v = u_1 u_2$ and $w = u_1 x x^{-1} u_2$ or $w = u_1 x^{-1} x u_2$.

Definition 2.7 (Free group). Let $\sim$ be the smallest equivalence relation on $W(X)$ such that

$$w \searrow v \implies w \sim v.$$

We define the **free group** on X to be

$$F(X) = W(X) / \sim.$$

Remark. This is what we formally mean by "products in $F(X)$ are distinct except for when they're obviously not".

Proposition 2.8 (Composition well-defined). Let $w_1 \sim v_1$ and $w_2 \sim v_2$ we have $w_1 w_2 \sim v_1 v_2$. Hence $[w_1][w_2] = [v_1][v_2]$.

Proof. Trivial. □

Theorem 2.9. $F(X)$ is a group.

Proof. We have

- Associativity: inherited from the monoid $W(X)$.
- Identity: $e = [\emptyset] \in W(X)$.
- Inverses: $[y_1, \dots, y_m] = [y_m^{-1}, \dots, y_1^{-1}]$ with $(y_i^{-1})^{-1} = y_i$.

□

2.1 Combinatorics of free groups

Definition 2.10 (Reduced Word). A word $w \in W(X)$ is **reduced** if $\neg \exists v \in W(X)$ such that $w \searrow v$.

Lemma 2.11 (Combinatorics of Free Groups). If $w \nearrow v \searrow u$ then either $w = u$ or $\exists v' \in W(X)$ s.t. $w \searrow v' \nearrow u$.

Proof. Suppose $w \searrow v$ removes letters $i, (i+1)$ of v , and $v \searrow u$ removes $j, (j+1)$ of u . We have the following cases:

- (i) $i = j \implies w = u$.
- (ii) $|i - j| = 1$. Then $\exists x \in X \cup X^{-1}, s, t \in W(X)$ s.t. $v = sxx^{-1}xt$, thus $w = u = sxt$.
- (iii) $|i - j| \geq 2$. Then $\exists x, y \in X \cup X^{-1}, r, s, t \in W(X)$ s.t. (WLOG)

$$\begin{aligned} v &= rxx^{-1}sy y^{-1}t \\ w &= rsyy^{-1}yy^{-1}t \\ u &= rxx^{-1}st., \end{aligned}$$

Set $v' = rst$. Then $w \searrow v' \nearrow u$.

□

Proposition 2.12 (Normality). $\forall w \in W(X) \exists! v \in W(X)$ reduced with $w \sim v$.

Proof. To prove existence, we define the length $l(y_1, y_2, \dots, y_m) = m$. If $w \searrow v$ then $l(v) \leq l(w) - 2 < l(w)$. Hence if $v \in [w]$ is of least length, then it is reduced.

For uniqueness, we note that if $w_1 \sim w_2$ are distinct and both reduced, then $w_1 \nearrow \dots \searrow w_2$, hence at some point in the sequence we will have a contraction followed by an expansion. Thus by using the Lemma 2.11 we can get $w_1 \searrow \dots \nearrow w_2$, which contradicts w_1, w_2 being reduced. □

Theorem 2.13 (Universal Property). Let X be a set, H a group and $\phi : X \rightarrow H$ function, then $\exists! \Phi : F(X) \rightarrow H$ homomorphism s.t. the diagram

$$\begin{array}{ccc} H & \xlongequal{\quad} & H \\ \uparrow \exists! \Phi & & \uparrow \phi \\ F(X) & \xleftarrow{i} & X \end{array}$$

commutes, where

$$\begin{aligned} i : X &\longrightarrow F(X) \\ x &\longmapsto [x]. \end{aligned}$$

Remark. Let G be any group $S \subseteq G, \langle S \rangle = G$, then any homomorphism $\Psi : G \rightarrow H$ uniquely determined by $\Psi|_S$. In general group homomorphism might not exist, since we may have some product of generators that is non-trivial in G but trivial in H . But since in free groups trivial elements are obviously so, we always have existence of the homomorphism.

Remark. Some authors may define directly free groups through universal property.

Proof of 2.13. Having defined

$$\begin{aligned} \tilde{\Phi} : W(X) &\longrightarrow H \\ x_1^{\varepsilon_1} \dots x_m^{\varepsilon_m} &\longmapsto \phi(x_1)^{\varepsilon_1} \dots \phi(x_m)^{\varepsilon_m}, \end{aligned}$$

with $x_i \in X, \varepsilon_i \in \{\pm 1\}$. This is a monoid homomorphism for which the UP holds as $\tilde{\Phi}(wv) = \tilde{\Phi}(w)\tilde{\Phi}(v)$. We need $w \sim v$ then $\tilde{\Phi}(w) = \tilde{\Phi}(v)$, so that $\tilde{\Phi}$ descends to a homomorphism $\Phi : F(X) \rightarrow H$.

Since $w \sim v$ is generated by contractions, suffices to show it for those. Have if $w \searrow v$ then $w = u_1 x^\varepsilon x^{-\varepsilon} u_2$ and $v = u_1 u_2$. Thus $\tilde{\Phi}(w) = \tilde{\Phi}(u_1)\tilde{\Phi}(x)^\varepsilon\tilde{\Phi}(x)^{-\varepsilon}\tilde{\Phi}(u_2) = \tilde{\Phi}(u_1)\tilde{\Phi}(u_2) = \tilde{\Phi}(v)$. $\square$

Definition 2.14 (Basis). A subset $Y \subset F(X)$ is a **basis** if the set injection $i : Y \hookrightarrow F(X)$ extends to an isomorphism $F(Y) \xrightarrow{\sim} F(X)$.

Remark. Note subtleties on changing categories.

Example. X is always a basis for $F(X)$.

Warning. Bases are not unique! For example, if $X = \{x, y\}, Y = \{x^{-1}, xy\}$, then Y is a basis for $F(X)$.

Proposition 2.15 (Base Cardinality). If Y is a basis for $F(X)$ then $|Y| = |X|$.

Proof. On ES#1, hint: use UP. $\square$

Definition 2.16 (Rank). The **rank** of a free group $F(X)$, is $\text{rk}F(X) = |X|$.

Remark. A **free group** is a group G isomorphic to $F(X)$ for some set X .

To offer another analogy with linear algebra, consider:

Theorem 2.17. Let $|X| < \infty$. If $Y \subseteq F(X)$ with $|Y| = |X|$ and $\langle Y \rangle = F(X)$ then Y is a basis for $F(X)$, thus spanning sets of same cardinality are bases.

Theorem 2.18 (Nielsen-Schreier). Any subgroup of a free group is free.

Proof. Omitted. $\square$

Remark. Unlike for vector spaces, rank of a proper subgroup of a free group can increase.

Lecture 3

2.2 More consequences of UP

Proposition 2.19. Every group G is a quotient Q of some free group $F(X)$. Moreover, if the group G is finitely generated we can take $|X| < \infty$, finite rank.

Proof. Let $\bar{X} \subseteq G$ be a generating set ($|\bar{X}| < \infty$). Let $\varphi : X \rightarrow \bar{X}$ be any surjection of sets. By UP, φ extends to $\tilde{\Phi} : F(X) \rightarrow G$. Since $\bar{X} \subseteq \text{im}(\varphi) \subseteq \text{im}(\tilde{\Phi})$, thus $\tilde{\Phi}$ is onto. $\square$

Corollary 2.20 (Meta-Theorem/Strategy). Suppose P is a property of groups preserved under quotients (e.g. if G has P and $N \triangleleft G$ then G/N has P) and there exists some $(k$ -generated) group without P , then no $(\text{rk} \geq k)$ free group has P .

Example. For $|X| \geq 2$, $F(X)$ is non-abelian.

We want to find a more effective description of the normal subgroups we're quotienting by:

Definition 2.21 (Normal Closure). Let G be a group, $S \subseteq G$ subset, the **normal closure** of S in G :

$$\langle\langle S \rangle\rangle^G := \bigcap_{S \subseteq N \triangleleft G} N.$$

Proposition 2.22. We have

- (i) $\langle\langle S \rangle\rangle^G \triangleleft G$.
- (ii) If $M \triangleleft G$ with $S \subseteq M$ then $\langle\langle S \rangle\rangle^G \leq M$.

Moreover, properties (i) and (ii) uniquely determine $\langle\langle S \rangle\rangle^G$.

Lemma 2.23. We have

$$\langle\langle S \rangle\rangle^G = \langle gsg^{-1} : s \in S, g \in G \rangle.$$

Proof. RHS satisfies (i) and (ii) of Proposition 2.22. □

Definition 2.24 (Group Presentation). A **group presentation** is a pair $\langle X|R \rangle$, where $R \subseteq F(X)$. Moreover, we denote by $\langle X|R \rangle$ the group

$$F(X) / \langle\langle R \rangle\rangle^{F(X)},$$

calling X the **generators** and R the **relations**. Given a group G we say $\langle X|R \rangle$ is a presentation for G if $G \cong \langle X|R \rangle$.

Warning. We'll use the notation X to denote both $X \subset F(X)$ and its image in $G \cong \langle X|R \rangle$. Moreover, for $w \in F(X)$, refer to w as an element of G , by identifying it with its image.

Proposition 2.25. Every group has a presentation.

Theorem 2.26. Let $G \cong \langle X|R \rangle$, H a group and $\varphi : X \rightarrow H$ a function. If $\widehat{\Phi} : F(X) \rightarrow H$ extends φ , then φ extends to $\Phi : G \rightarrow H$ if and only if $R \subseteq \ker(\widehat{\Phi})$.

Proof.

$$\begin{array}{ccc}
 F(X) & \xrightarrow{\hat{\Phi}} & H \\
 \downarrow \pi & \nearrow \exists! \Phi & \\
 G = F(X) / \langle\langle R \rangle\rangle^{F(X)} & &
 \end{array}$$

By Proposition 2.22, we have

$$R \subseteq \ker(\hat{\Phi}) \iff \langle\langle R \rangle\rangle^{F(X)} \subseteq \ker(\hat{\Phi}),$$

and we can now use the general fact about normal subgroups that, given $N \triangleleft \hat{G}$, $\hat{\Psi} : \hat{G} \rightarrow H$, $\hat{\Psi}$ descends to $\Psi : \hat{G}/N \rightarrow H$ iff $N \leq \ker(\hat{\Psi})$, given by $g \cdot N \mapsto \hat{\Psi}(g)$. $\square$

Remark. As usual, if $\Phi : G \rightarrow H$ exists, then it is unique.

Remark. This result is actually useful in practice. Moreover, it is also a good way to check if something is a presentation for G . Fairly standard strategy to try to build isomorphisms.

Example 2.27 (Finite dihedral groups). We wish to show that $D_{2n} \cong \langle X|R \rangle$, where $X = \{x, y\}$, $R = \{x^n, y^2, xyxy\}$. Recall that $|D_{2n}| = 2n$, $D_{2n} = \langle r, s \rangle$ where r is rotation through $\frac{2\pi}{n}$ and s reflection, satisfying

$$r^n = s^2 = (rs)^2 = e.$$

By Theorem 2.26, we have a well-defined surjective homomorphism

$$\begin{aligned}
 \Phi : \langle X|R \rangle &\longrightarrow D_{2n} \\
 x &\longmapsto r \\
 y &\longmapsto s.
 \end{aligned}$$

We now have to hope relations are sufficient, otherwise argument would have worked with any set of relations that hold in D_{2n} (including the empty one).

To prove Φ is injective, consider that in $\langle X|R \rangle$, we have

- (i) $yx = x^{-1}y \iff y^{-1}xyx = (y^2)^{-1} \cdot y(xyxy)y^{-1} \in \langle\langle R \rangle\rangle^{F(X)}$.
- (ii) $yx^{-1} = xy$, since $x^{-1}(xy)x = x^{-1}(yx^1)x$, which gives $yx = x^{-1}y$, by (i).
- (iii) $y = y^{-1}$, since $y^2 \in R$.

So we have injectivity. Now let $g \in \langle X|R \rangle$, be represented by $w \in F(X)$. By using (iii), we can assume that w contains no y^{-1} and by induction can assume $w = x^i y^j$ with $i, j \in \mathbb{Z}$. Moreover, since $x^n, y^2 \in R$, we can assume $0 \leq i < n, j \in \{0, 1\}$.

This implies that $|\langle X|R \rangle| \leq 2n = |D_{2n}|$.

Remark. Previous proof is construing some normal form. But it is the same thing we do all the time while working in dihedral groups.

Remark. Given a group G , to prove that a $\langle X|R \rangle$ is a presentation for G , we define a function $X \rightarrow G$ using Theorem 2.26, then checking that it extends to homomorphism $\langle X|R \rangle \rightarrow G$ is easy, and but proving Φ is injective is hard, in general. In fact it is an open problem for quite a lot of finite groups.

Remark. We can succeed in proving presentation if the elements of G admit a **normal form**.

Remark. If don't have normal form, there are geometric methods to prove injectivity. We can for instance try to find some geometric object on which the group acts.

Definition 2.28 (Finitely Presented). We say G is **finitely presented** or **finitely presentable** if there exists a presentation $\langle X|R \rangle$ for G with $|X| < \infty, |R| < \infty$.

Remark. We will prove that not all finitely generated groups are finitely presented. A quick argument is that finitely presented groups are countable, so it suffices to prove that finitely generated groups are uncountable.

Exercise (ES#1.7). Let $N \triangleleft G$. If N and G/N are finitely presented, so is G .

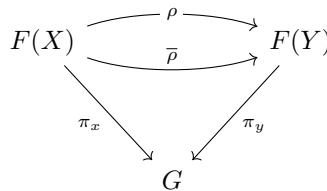
Proposition 2.29. Suppose that G is finitely presented and let $\langle X|R \rangle$ be a presentation for G with $|X| < \infty$ (but not necessarily $|R| < \infty$). Then finite $\exists R_0 \subseteq R$ s.t. $\langle X|R_0 \rangle$ is also a presentation for G .

Proof. We have

- (i) Suppose that $\langle \langle R \rangle \rangle^{F(X)}$ is *finitely normally generated*, i.e. there exists finite $Q \subseteq \langle \langle R \rangle \rangle^{F(X)}$ such that $\langle \langle Q \rangle \rangle^{F(X)} = \langle \langle R \rangle \rangle^{F(X)}$. For then, $\forall q \in Q \exists R_q \subseteq R$ finite s.t. $q \in \langle \langle R_q \rangle \rangle^{F(X)}$.
- (ii) Set $R_0 = \bigcap_{q \in Q} R_q$. Then

$$\langle \langle Q \rangle \rangle^{F(X)} \subseteq \langle \langle R_0 \rangle \rangle^{F(X)}.$$

Let $\langle Y|S \rangle$ finite presentation for G .



for $x \in X$, let $v_x \in F(Y)$ with $\pi_y(v_x) = \pi_x(x)$, and similarly for $y \in Y$, let $v_y \in F(X)$ with $\pi_x(v_y) = \pi_y(y)$. Define

$$\begin{aligned}
 \phi : F(X) &\longrightarrow F(Y) \\
 x &\longmapsto v_x,
 \end{aligned}$$

and similarly $\bar{\rho} : F(Y) \rightarrow F(X)$.

We claim that $Q = \bar{\rho}(S) \cup T$ normally generates $\langle\langle R \rangle\rangle^{F(X)} = \ker(\pi_x)$ where $T = \{\bar{\rho}(x)x^{-1} : x \in X\}$.

Certainly we have $\bar{\rho}(S), T \subseteq \ker(\pi_x)$, conversely, for $w \in \ker(\pi_x)$ we have $\rho(w) \in \ker(\pi_y) = \langle\langle S \rangle\rangle^{F(Y)}$. and thus $\bar{\rho}(\rho(w)) \in \langle\langle \bar{\rho}(S) \rangle\rangle^{F(X)}$.

Now $w = (\bar{\rho}(\rho(w)) \cdot w^{-1})^{-1} \bar{\rho}(\rho(w))$ hence $\bar{\rho}(\rho(w)) \cdot w^{-1} \in \langle\langle T \rangle\rangle^{F(X)}$. $\square$

Remark. So finite presentability acts a bit like compactness, as we can pass from open covers to finite subcovers. This is actually more than coincidence as finitely presented groups arise as fundamental groups of compact topological spaces.

Compare with result for finitely generated groups:

Proposition 2.30. Let G be finitely generated and $G = \langle Y \rangle$, then $\exists Z \subseteq Y$ finite such that $G = \langle Z \rangle$.

Lecture 4

3 Word Problems

Reading finite presentations is not easy at all.

Example. In the following groups

$$G_1 = \langle a, b, c \mid aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle,$$

$$G_2 = \langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle,$$

one is trivial, and the other one is a candidate for non-sofic group. It is hard to tell which is which at first glance.

Question. Given $v, w \in F(X)$, $|X| < \infty$, do v and w represent the same element of $G = \langle X \mid R \rangle$ finitely presented (notation $v \cdot w^{-1} =_G e$)?

Definition 3.1 (Solvable Word Problem). $G = \langle X \mid R \rangle$ has **solvable word problem** if there is an algorithm which takes as input $w \in F(X)$ and determines whether or not $w =_G e$.

Remark. Note that

- (i) We don't precisely define "algorithm".
- (ii) The algorithm is tailor made for a given presentation. So the algorithm *knows* the presentation. We don't ask for a generic algorithm, which cannot exist.
- (iii) We're yet to show that solvability of word problems presentation independent.

Remark. There are two parts of the word problem:

- (i) Deciding that $w =_G e$,

- (ii) Deciding that $w \neq_G e$.

Proposition 3.2. For any $G = \langle X | R \rangle$ f.p. then there is an algorithm $\text{TRIVIAL}(X, R)$ which takes as input $w \in W(X)$ and

- (i) Answers YES if $[w] =_G e$
- (ii) Does not terminate if $[w] \neq_G e$.

Proof. We do

- (i) Put w in reduced form by deleting all $x \cdot x^{-1}$ and $x^{-1} \cdot x$.
- (ii) Produce a list $(w_n)_n$ of words $w_n \in W(X)$ representing all elements of $\langle \langle R \rangle \rangle^{F(X)}$, which is not just countable but also enumerable since $|X| < \infty$. For example,

$$(v_1 r v_1^{-1})^{\varepsilon_1} \cdots (v_R r v_R^{-1})^{\varepsilon_R} \text{ for } r_i \in R, v_i \in W(X), \varepsilon_i \in \{\pm 1\}.$$

These are ordered in both increasing in R and $\max_i l(v_i)$. Choice of enumeration does not matter, any will do.

- (iii) For each of w_n , put w_n in reduced form and compare w and w_n letter by letter.
- (iv) If w and w_n agree, output YES. If not, move to w_{n+1} and apply (iii).

□

Remark. This is a parallel stream/coinductive algorithm, step (ii) is actually infinite. Computers can only do one thing at a time but they can divide attention among finite tasks. In particular, any computer can run an arbitrary finite number of computers, so even if one of the inner computers is not going to halt, the output of a second inner computer may be produced.

Corollary 3.3. The word problem is always semi-decidable, so to solve WP suffices to find an algorithm verifying $w \neq_G e$.

Examples of solvable word problem:

Example. Finite groups have decidable word problem.

Example. Finitely generated abelian groups have decidable word problem.

Both on ES#1.

Example. Every finite rank free group has decidable word problem, as algorithm will always terminate, since

$$F(X) = \langle X | \emptyset \rangle.$$

so put $w \in W(X)$ in reduced form and if $w \neq \emptyset, w \neq_{F(X)} e$.

Remark. If we have an algorithm for one presentation, and we have another presentation, we can algorithmically build an isomorphism and then pass the word problem to first one.

Exercise. Prove that $G = \langle x, y | x^{-1}y^{-1}xy \rangle \cong \mathbb{Z}^2$ via $x \mapsto (1, 0), y \mapsto (0, 1)$, with similar argument to Example 2.27.

Proposition 3.4. Given $w \in W(X)$, there exists unique $(a, b) \in \mathbb{Z}^2$ such that

$$W_{a,b} = W \cdot x^a \cdot y^{-b} \in \langle \langle x^{-1}y^{-1}xy \rangle \rangle^{F(X)},$$

thus we can apply $\text{TRIVIAL}(X, \{x^{-1}y^{-1}xy\})$ to each $W_{a,b}$ and if it returns YES for $(a, b) \neq (0, 0)$ then $w \neq_G e$.

Remark. For two generators, can run through word and keep count of x, y .

Theorem 3.5 (Novikov-Boone '50s). There exists finitely presented groups with unsolvable word problem.

Proof. Outside scope of course, see [AL12] and lecture notes for old Part III LT17 course *Decision Problems in Group Theory*. $\square$

Remark. There was Part III course on building up the result.

The proof tries to embed a Turing machine inside a group and prove that WP would imply a solution to the halting problem. It is considerably easier to prove undecidability of word problems for monoids, done by Turing.

There are two independent proofs by the two authors, due to cold war re-discoveries. The general strategy is the same but technical decisions are different.

Proposition 3.6. There exists an algorithm HOMS which takes as input two finite presentations

$$G = \langle X | R \rangle \quad H = \langle Y | S \rangle,$$

and outputs a list of all homomorphisms $\Phi : G \rightarrow H$.

Proof. It suffices to output a list of all functions $\varphi : X \rightarrow F(Y)$ such that there exists a homomorphism $\Phi : G \rightarrow H$ with $\Phi(x) =_H \varphi(x)$. We do

- (i) List of all functions $\varphi_n : X \rightarrow F(Y)$,
- (ii) For each n , compute $\tilde{\Phi}_n(r) \in F(Y)$, and apply $\text{TRIVIAL}(Y, S)$ to $\tilde{\Phi}_n(r)$,
- (iii) If YES to all, then $\tilde{\Phi}_n$ induces $\Phi_n : G \rightarrow H$ by Theorem 2.26 so we output φ_n .

$\square$

Remark. Another parallel algorithm as need to handle TRIVIAL not terminating.

Remark. This list may contain duplicates and you can only tell if the word problem is solvable.

Lecture 5

Remark. It is an active area of research to find faster algorithms (in theory and in practice) for word problems for groups.

Corollary 3.7. There is an algorithm ISO which takes as input two finite presentations

$$\begin{aligned} G &= \langle X | R \rangle \\ H &= \langle Y | S \rangle, \end{aligned}$$

and if $G \cong H$ outputs mutually inverse homomorphisms

$$\begin{aligned} \Phi : G &\rightarrow H, \\ \Psi : G &\rightarrow H, \end{aligned}$$

and does not terminate if $G \not\cong H$.

Proof. We do

- (i) Apply HOMS to G and H , in both directions,
- (ii) For each $\varphi_n : X \rightarrow F(Y), \psi_m : Y \rightarrow F(X)$ on our list inducing homomorphisms

$$\begin{aligned} \widehat{\Phi}_n &: F(X) \rightarrow F(Y), \\ \widehat{\Phi}_m &: F(Y) \rightarrow F(X), \end{aligned}$$

we apply TRIVIAL(x, R) to $\widehat{\Phi}_m(\varphi_n(x)) \cdot x^{-1}$ for each $x \in X$. If "YES" for all of them, then $\Psi_m \circ \Phi_n$ is the identity on X hence on G , and likewise for $\Phi_n \circ \Psi_m$, Y and H .

□

Corollary 3.8. If $\langle X | R \rangle \cong \langle Y | S \rangle$, then $\langle X | R \rangle$ has solvable WP iff $\langle Y | S \rangle$ does.

Proof. Suppose SOLVER(Y, S) solves the WP. By Corollary 3.7, we can assume to have $\varphi : X \rightarrow F(Y)$ such that $\widehat{\Phi} : F(X) \rightarrow F(Y)$ induces an isomorphism

$$\Phi : \langle X | R \rangle \xrightarrow{\sim} \langle Y | S \rangle.$$

For $w \in F(X)$, we apply SOLVER(Y, S) to $\widetilde{w} \in F(Y)$.

□

Remark. This does not mean that the word problem is as hard to solve for all presentations. You will often want to work with a particularly nice presentation.

Question. Are there natural conditions you can put on a group to make the word problem not decidable?

Answer. There are universal finitely presented groups, in which every other finitely presentable group may be embedded. By Novikov-Boone, any universal finitely presented group must have undecidable word problem. See [Val77, Bir03] and further references [BH74, Dav77, NS16] for more.

Remark. There is an open competition to find the smallest group presentation with undecidable word problem. The current record stands at 12 relators.

4 Free Products

Definition 4.1 (Free Product). Let $G = \langle X | R \rangle$, $H = \langle Y | S \rangle$, with $X \cap Y = \emptyset$. The **free product** of G and H is

$$G * H = \langle X \cup Y | R \cup S \rangle.$$

Remark. Free products are associative and commutative.

Example. We have $\mathbb{Z} \cong \langle \{x\} | \emptyset \rangle$, so for $|X| = n$, have

$$F(X) \cong \mathbb{Z} * \cdots * \mathbb{Z},$$

the n -fold free product of $\mathbb{Z}$.

Example (ES#1). We have $D_\infty \cong C_2 * C_2$, see discussion in Coxeter groups.

Theorem 4.2 (UP for Free Products). There exists injective homomorphisms

$$G \xrightarrow{i_G} G * H \xleftarrow{i_H} H$$

such that for all groups K and homomorphisms Φ_G, Φ_H , there exists a unique homomorphism $\Phi : G * H \rightarrow K$ such that the following diagram commutes

$$\begin{array}{ccccc} G & \xrightarrow{i_G} & G * H & \xleftarrow{i_H} & H \\ & \searrow \Phi_G & \downarrow \exists! \Phi & \swarrow \Phi_H & \\ & & K & & \end{array}$$

Proof. We have

$$\begin{aligned} \tilde{i}_G : F(X) &\longrightarrow G * H \\ x &\longmapsto x, \end{aligned}$$

descends to $i_G : G \rightarrow G * H$, by Theorem 2.26. We now define

$$\begin{aligned}\tilde{\Psi} : F(X \amalg Y) &\longrightarrow K \\ x &\longmapsto \Phi_G(x) \\ y &\longmapsto \Phi_H(y),\end{aligned}$$

thus for $r \in R$, $\tilde{\Psi}(r) = \Phi_G(r) = e_K$ and likewise for $s \in S$, $\tilde{\Psi}(s) = e_K$, so $\tilde{\Psi}$ descends to $\Psi : G * H \rightarrow K$, again by Theorem 2.26.

Moreover, $(\Psi \circ i_G)(x) = \Psi_G(x)$ by construction, so since $G = \langle X | R \rangle$, $\Psi \circ i_G = \Phi_G$ and likewise for $\Psi \circ i_H = \Phi_H$. This gives existence.

For uniqueness, observe that if $\Psi' : G * H \rightarrow K$ also satisfied the commutative diagram, then

$$\begin{aligned}\Psi'(x) &= (\Psi' \circ i_G)(x) \\ &= \Phi_G(x) \\ &= (\Psi \circ i_G)(x),\end{aligned}$$

thus we have $\Psi'|_X = \Psi|_X$ and likewise $\Psi'|_Y = \Psi|_Y$ and $G * H = \langle X \cup Y | R \cap S \rangle$, so $\Psi' = \Psi$.

Remains to prove injectivity, but since $\exists! \Psi : G * H \rightarrow K$ such that

$$\begin{aligned}\forall h \in H, (\Psi \circ i_H)(h) &= e \\ \forall g \in G, (\Psi \circ i_G)(g) &= g,\end{aligned}$$

we deduce that i_G and i_H are injective. $\square$

Remark. We'll identify G, H with $i_G(G), i_H(H) \leq G * H$, writing $\langle G \cup H \rangle$ for $G * H$.

Proposition 4.3. Suppose $G \otimes H$ is a group with injective homomorphisms

$$G \xrightarrow{i_G} G \otimes H \xleftarrow{i_H} H$$

satisfying the universal property. Then $G \otimes H \cong G * H$.

Proof. Standard uniqueness of UP proof. Apply both UPs. $\square$

Remark. This proves independence of free products from presentations and also that the isomorphisms commute with the embeddings.

Exercise. Any group satisfying the UP for free groups on X is isomorphic to $F(X)$.

Definition 4.4 (Reduced word). Let G, H be non-trivial groups (assuming disjoint underlying sets). A **reduced word** in G and H is a finite string of symbols

$$y_1 y_2 \cdots y_m,$$

where $y_i \in (G \setminus \{e\}) \amalg (H \setminus \{e\})$, and, for all $1 \leq i \leq m-1$, either $y_i \in G, y_{i+1} \in H$ or $y_i \in H, y_{i+1} \in G$. We consider the empty word $\emptyset$ to be a reduced word.

We will write $R(G, H)$ for the **set of reduced words** in G and H .

Remark. We'll show that uniqueness of reduced words precisely characterises a free product.

Lecture 6

Remark. If $G, H \leq K$, then every reduced word $w \in R(G, H)$ represents a unique element $\bar{w} \in K$. Conversely if $\langle G, H \rangle = K$, every element of K is represented by a reduced word $w \in R(G, H)$. This expression may not in general be unique!

Theorem 4.5 (Uniqueness of Reduced Words). We have

- (i) Every $\emptyset \neq w \in R(G, H)$ represents a non-trivial element of $G * H$.
- (ii) Conversely, if $G, H \leq K$, and every non-empty reduced word $w \in R(G, H)$ represents a non-trivial element of K , then the homomorphism

$$\begin{aligned} I : G * H &\longrightarrow \langle G, H \rangle \\ g &\longmapsto g \\ h &\longmapsto h, \end{aligned}$$

is an isomorphism.

Proof. First, we note that this homomorphism is well-defined by UP of free products. For (i), to find an action $G \curvearrowright R(G, H)$, we define the homomorphism

$$\rho_G : G \longrightarrow \text{Symm}(R(G, H)),$$

as

$$\rho_G(g) = \begin{cases} \emptyset & \longmapsto g \\ g^{-1} \cdot w & \longmapsto w \\ g' \cdot w & \longmapsto (gg') \cdot w \\ h \cdot w & \longmapsto g \cdot h \cdot w \end{cases}$$

where $\emptyset \neq w \in R(G, H)$, $g, g' \in G \setminus \{e\}$ and $h \in H \setminus \{e\}$ and $g' \neq g^{-1}$ and likewise for $\rho_H : H \rightarrow \text{Symm}(R(G, H))$. By some tedious bookkeeping can show that these are well-defined actions.

By UP for Free Products, we obtain

$$\Psi : G * H \rightarrow \text{Symm}(R(G, H)).$$

Thus if $\bar{w} \in G * H$ is represented by $\emptyset \neq w \in R(G, H)$, then by construction $\Psi(\bar{w})(\emptyset) = w \neq \emptyset$, thus $\Psi(\bar{w}) \neq \text{id}$, $\bar{w} \neq_{G * H} e$.

For (ii), I is a well-defined surjective homomorphism by UP. Suffices to show I is injective thus let $e \neq \bar{w} \in G * H$ be represented by $\emptyset \neq w \in R(G, H)$. But w also represents $I(\bar{w}) \in \langle G, H \rangle$ so by assumption $I(\bar{w}) \neq e$. $\square$

Remark. We will now show that we can sometimes check this property.

Lemma 4.6 (Ping-Pong for Free Products). Let K be a group, non-trivial $G, H \leq K$, with $|G| \geq 3$. Suppose K acts on a set X and exists $\emptyset \neq A, B \subseteq X$ such that

- (i) $\forall g \in G \setminus \{e\}, g(B) \subseteq A$,
- (ii) $\forall h \in H \setminus \{e\}, h(A) \subseteq B$,
- (iii) $A \not\subseteq B$,

then

$$\begin{aligned} I : G * H &\longrightarrow \langle G, H \rangle \\ g &\longmapsto g \\ h &\longmapsto h, \end{aligned}$$

is an isomorphism.

Remark. We may think of A and B as sides of a ping-pong table.

Proof. We wish to check conditions of Theorem 4.5(ii) hold. Let $\emptyset \neq w \in R(G, H)$. Four cases to check, depending on first and last letter of w :

- (i) $w = g_1 h_1 \cdots g_m h_m g$, with $g_i, g \in G \setminus \{e\}, h_i \in H \setminus \{e\}$,

$$\begin{aligned} \bar{w}(B) &= (g_1 h_1 \cdots g_m h_m g)(B) \\ &\subseteq (g_1 h_1 \cdots g_m h_m)(A) \\ &\subseteq (g_1 h_1 \cdots g_m)(B) \\ &\vdots \\ &\subseteq (g_1 h_1)(A) \\ &\subseteq g_1(B) \\ &\subseteq A \not\subseteq B, \end{aligned}$$

if $\bar{w} \neq_K e$.

- (ii) $w = g_1 h_1 \cdots g_m h_m$. Let $g \in G \setminus \{e\}$ be such that $g \neq g_1$, which is possible as $|G| \geq 3$. Then $g^{-1} \bar{w} g = (g^{-1} g_1) h_1 \cdots g_m h_m g \neq_K e$ by (i). Hence $\bar{w} \neq_K e$ also.

- (iii) $w = h_1 g_1 \cdots h_m g_m$, then

$$\bar{w} = g_m^{-1} h_m^{-1} \cdots g_1^{-1} h_1^{-1} \neq_K e,$$

by (ii), so $\bar{w} \neq_K e$.

- (iv) $w = h g_1 h_1 \cdots g_m h_m$, $h \in H \setminus \{e\}$. Then pick some $g \in G \setminus \{e\}$ and observe that

$$g^{-1} \bar{w} g = g^{-1} h g_1 h_1 \cdots g_m h_m g \neq_K e,$$

by (i), hence $\bar{w} \neq_K e$

□

Corollary 4.7 (Ping-Pong for Free Groups). Suppose K acts on X , $\emptyset \neq A, B \subseteq X$, and there exist $g, h \in K$ such that

- (i) $A \not\subseteq B, B \not\subseteq A$,
- (ii) $\forall m \in \mathbb{Z} \setminus \{0\}, g^m(B) \subseteq A, h^m(A) \subseteq B$.

Then $\langle g, h \rangle \leq K$ is a free group of rank 2, and $\{g, h\}$ is a basis.

Proof. By (i) and (ii), g, h have infinite order in K , so $G := \langle g \rangle, H := \langle h \rangle \cong \mathbb{Z}$. Applying Lemma 4.6, $\langle g, h \rangle \cong G * H = \langle g, h | \emptyset \rangle \cong F(\{g, h\})$ $\square$

Remark. Some Geometric Group Theory references will call this weaker result the Ping-Pong Lemma.

Example (ES#2). Let $g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} h = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then $\langle g, h \rangle$ is a free group of rank 2.

Lecture 7

5 Soluble Groups

Definition 5.1 (Soluble). A group G is **soluble** if it has an **abelian series**, i.e.

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_d = \{e\},$$

such that G_i/G_{i+1} is abelian. The **derived length** of G is the minimal possible d .

Example 5.2 (Abelian Groups). Abelian groups are precisely the soluble groups of derived length 1.

Example 5.3 (Dihedral groups). D_{2n}, D_∞ have derived length 2.

Example (S_4). S_4 has derived length 3.

Exercise (UT Matrices, ES#2). If k is a field, then

$$B_n(k) = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_n \end{pmatrix} : a_i \in k^\times \right\} \leq \mathrm{GL}_n(k),$$

is soluble and the derived length grows as function of n .

Example 5.4 (Non). Every non-abelian simple group is non-soluble, e.g. A_5 .

Proposition 5.5 (Soluble Closed). Let G be a group, $H \leq G$ and $N \triangleleft G$, then

- (i) If G is soluble of d.l. $\leq d$, then H is soluble of d.l. $\leq d$,
- (ii) If G is soluble of d.l. $\leq d$, then G/N is soluble of d.l. $\leq d$,

(iii) If $N, G/N$ are soluble, resp. of d.l. $\leq d, \leq e$, then G is soluble of d.l. $\leq d + e$.

Proof. We have

- (i) Let $(G_i)_{i=0}^d$ be an abelian series for G . We wish to show that $(G_i \cap H)_{i=0}^d$ is an abelian series for H . By the Second Isomorphism Theorem, we have

$$G_i \cap H / G_{i+1} \cap H \cong (G_i \cap H)G_{i+1} / G_{i+1} \leq G_i / G_{i+1},$$

which is abelian by assumption.

- (ii) We wish to show that $(G_i N / N)_{i=0}^n$ is an abelian series for G/N . By the Second and Third Isomorphism Theorems, we have

$$\begin{aligned} (G_i N / N) / (G_{i+1} N / N) &\cong G_i N / G_{i+1} N \\ &= G_i (G_{i+1} N) / G_{i+1} N \\ &\cong G_i / (G_i \cap (G_{i+1} N)), \end{aligned}$$

but $G_{i+1} \leq G_i \cap (G_{i+1} N)$, so we have a quotient of G_i / G_{i+1} .

- (iii) Let $(N_i)_{i=0}^d, (Q_i)_{i=0}^e$ be abelian series for $N, G/N$. Define

$$\begin{aligned} \pi : G &\longrightarrow G/N \\ g &\longmapsto gN, \end{aligned}$$

take $G_i = \pi^{-1}(Q_i)$, then get $G_0 = G, G_i \triangleright G_{i+1}, G_e = N, G_i / N = Q_i$. This gives

$$G_i / G_{i+1} \cong Q_i / Q_{i+1},$$

by the Third Isomorphism Theorem, so we have

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_e = N = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_d = \{e\}.$$

□

Definition 5.6 (Commutator). Let G be a group, $g, h \in G$, $[g, h] = g^{-1}h^{-1}gh$ is the **commutator** of g and h .

For $X, Y \subseteq G$, we define the **commutator subgroup**

$$[X, Y] = \langle [x, y] : x \in X, y \in Y \rangle.$$

Remark. We have $[g, h] = e \iff gh = hg \in G$

Lemma 5.7. We have

- (i) If $M, N \triangleleft G$, then $[M, N] \triangleleft G$,
- (ii) $G/[G, G]$ is abelian.
- (iii) If $N \triangleleft G$, and G/N is abelian, then $[G, G] \leq N$.

Proof. We have

- (i) For $g \in G, m \in M, n \in N$, we have

$$g[m, n]g^{-1} = [gmg^{-1}, gng^{-1}] \in [M, N],$$

(ii) For $g, h \in G$, and

$$[gN, hN] = [g, h]N,$$

$$\text{so } G/N \text{ is abelian} \iff \forall g, h \in G, [g, h] \in N \iff [G, G] \leq N.$$

□

Definition 5.8 (Abelianization). Let G be a group, we call $G_{\text{ab}} = G/[G, G]$ the **abelianization** of G .

Corollary 5.9. By Lemma 5.7, G is the unique maximal abelian quotient of G .

Question. How do you compute derived lengths in practice?

Answer. The best abelian series should be given by repeatedly taking abelianizations.

Definition 5.10 (Derived Series). The **derived series** $(G^{(i)})_i$ is given by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

Proposition 5.11. We have $\forall i, G^{(i)} \triangleleft G$.

Proof. Induction on Lemma 5.7. □

Theorem 5.12. If G is soluble (of derived length d), then $(G^{(i)})$ is an abelian series for G and $G^{(d)} = 0$.

Proof. We have $G^{(i)}/G^{(i+1)} = G_{\text{ab}}^{(i)}$, which shows that the derived series is an abelian series provided it reaches the identity.

Let $(G_i)_{i=0}^d$ be an abelian series for G with $G_d = \{e\}$. We will prove that $G^{(i)} \leq G_i$ by induction on i . Certainly $G^{(0)} = G = G_0$, so assume now that G is normal over $G^{(i)} \leq G_i$, then

$$G^{(i+1)} = [G^{(i)}, G^{(i)}] \leq [G_i, G_i].$$

Since G_i/G_{i+1} is abelian, so by Lemma 5.7, we have $[G_i, G_i] \leq G_{i+1}$. □

Remark. This actually shows that the derived series is the fastest shrinking even among all derived series of derived length.

Example (D_8). $D_8 = \langle s, r \rangle$, we have

$$D_8 \triangleright \langle r \rangle \cong C_4 \triangleright \{e\},$$

but also

$$D_8 \triangleright \langle r \rangle \triangleright \langle r^2 \rangle \triangleright \{e\}.$$

Remark. We will prove later that every soluble group is sofic, as a special case of Theorem 8.9.

Definition 5.13 (Torsion). A group G is **torsion** if every element has finite order.

Example. Finite groups are torsion.

Example. In the group $\mathbb{Q}/\mathbb{Z}$, each element has $\text{ord}(\frac{p}{q} + \mathbb{Z})|q$, so it is torsion.

Question 5.14 (Burnside 1902). Is every finitely generated torsion group finite?

Answer. No. We'll construct a counter-example in Section 7.3.

Question (Restricted Burnside Problem). Is every finitely generated torsion group of exponent n finite?

Answer. No, proved by Novikov and Adian [Iva94, Adi15].

Theorem 5.15. Every finitely generated torsion soluble group is finite.

Proof. See ES#2. □

Lecture 8

5.1 Nilpotent Groups

Definition 5.16 (Nilpotent Group). A **central series** for the group G is

$$G = G_1 \geq G_2 \geq \cdots \geq G_{c+1} = e,$$

such that $\forall i, [G, G_i] \leq G_{i+1}$. We call c the **length** of the series.

Remark. Universal convention to have different indexing for central and abelian series.

Definition 5.17 (Nilpotency Class). G is **nilpotent of class** $\leq c$ if it has a central series of length $\leq c$.

Proposition 5.18 (Nilpotent Soluble). G is nilpotent of class c . Then

- (i) G is soluble (of d.l. $\leq c$),
- (ii) Any subgroup or quotient of G is nilpotent (of class $\leq c$).

Proof. We have

(i) Any central series is abelian since $[g, n] = (g^{-1}n^{-1}g) \cdot n$ gives

$$\forall i, [G, G_i] \leq G_{i+1} \iff G_i \triangleleft G \text{ and } G_{i-1}/G_i \leq Z(G/G_i).$$

(ii) Mutatis-mutandis of Proposition 5.5.

□

Remark. Notice that we have no result here for extensions.

Example (Abelian Nilpotent). Abelian groups are nilpotent of class 1.

Exercise. Every finite p -group is nilpotent (ES#2).

Exercise (ES#2). If k is a field then

$$U_n(k) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} : a_i \in k^\times \right\} \leq \text{GL}_n(k),$$

is nilpotent.

Example (Non). Any non-soluble group.

Definition 5.19 (Lower Central Series). The **lower central series** for G is defined for $i \in \mathbb{N}_{\geq 0}$ by

$$\gamma_{i+1}(G) = \begin{cases} G & \text{if } i = 0 \\ [G, \gamma_i(G)] & \text{otherwise} \end{cases}$$

Proposition 5.20 (Lower Central Series is Central). If G is nilpotent of class c , then the $(\gamma_i(G))_{i=1}^{c+1}$ is a central series for G , and $\gamma_{c+1}(G) = \{e\}$.

Proof. Let $(G_i)_{i=1}^{c+1}$ be a central series for G . By construction, $(\gamma_i(G))_{i=1}^{c+1}$ satisfies the finite descending chain of subgroups condition, so it suffices to show that $\forall i, \gamma_i(G) \leq G_i$. We have

$$\gamma_{i+1}(G) = [G, \gamma_i(G)] \leq [G, G_i] \leq G_{i+1},$$

so claim follows by induction.

□

Example 5.21 (Dihedral Generators). For n odd, $D_{2n} = \langle r, s \rangle$, since $\gamma_2(D_{2n}) = \gamma_3(D_{2n}) = \langle r \rangle$.

Example (Soluble Non-Nilpotent). We have for

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in k^\times, b \in k \right\} \leq \text{GL}_2(k),$$

then G has derived length 2, but for $|k| \geq 3$, we have

$$\gamma_2(G) = \gamma_3(G) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in k \right\}.$$

5.2 Relating Class and Derived Length

Lemma 5.22 (Commutator Identities). Let G be a group, $g, h, k \in G$, then

- (i) $[g, h]^{-1} = [h, g]$
- (ii) $k[g, h]k^{-1} = [kgk^{-1}, khk^{-1}]$
- (iii) $[g, h^{-1}] = h[h, g]h^{-1} = [h, g] \cdot [[h, g], h^{-1}]$
- (iv) $[g, hk] = [g, k]k^{-1}[g, h]k = [g, k][g, h][[g, h], k]$
- (v) $[gh, k] = h^{-1}[g, k]h[h, k] = [g, k][[g, k], h][h, k]$
- (vi) $(h^{-1}[[g, h^{-1}], k]h)(k^{-1}[[h, k^{-1}], g]k)(g^{-1}[[k, g^{-1}], h]g) = e$.

Where the (vi) is the **Hall-Witt Identity**.

Proof. Direct computation. □

Remark. Some vague analogy between group-theoretic commutator and Lie Brackets. The Hall-Witt identity is a non-abelian group analogue of Jacobi identity, has to do with local *flattening* or linearising.

Lemma 5.23 (Three Subgroups). Let $X, Y, Z \leq G$, $N \triangleleft G$. Then

$$[[X, Y], Z], [[Y, Z], X] \leq N \implies [[Z, X], Y] \leq N.$$

Proof. By Lemma 5.22(iv), $[[Z, X], Y]$ is generated by conjugates of elements of the form $[[z, x^{-1}], y]$. Moreover, we know that $[[x, y^{-1}], z], [[y, z^{-1}], x] \in N$, so by Hall-Witt and normality of N , we have $[[z, x^{-1}], y] \in N$ also. □

Theorem 5.24. Let $m, n \geq 1$. Then

$$[\gamma_n(G), \gamma_m(G)] \leq \gamma_{m+n}(G).$$

Proof. Induction on n . True for $n = 1$ for all m by definition of central series. Suppose true for m and all m .

$$\begin{aligned} [\gamma_{n+1}(G), \gamma_m(G)] &= [[G, \gamma_n(G)], \gamma_m(G)] \\ [[\gamma_n(G), \gamma_m(G)], G] &\leq [\gamma_{n+m}(G), G] \\ &= \gamma_{n+m+1}(G), \\ [[\gamma_m(G), G], \gamma_n(G)] &= [[G, \gamma_m(G)], \gamma_n(G)] \\ &= [\gamma_{m+1}(G), \gamma_n(G)] \\ &= [\gamma_n(G), \gamma_{m+1}(G)] \\ &\leq \gamma_{n+m+1}(G). \end{aligned}$$

Thus the result follows by the Three Subgroup Lemma 5.23. □

Remark. This justifies why central series start at 0.

Corollary 5.25. If G is nilpotent of class c , then G is soluble of d.l. $\leq \log_2 c + 1$.

Proof. $(G^{(i)})$ derived series of G , by Theorem 5.24 and induction

$$\forall i. G^{(i)} \leq \gamma_2(G).$$

□

Remark. The bound is the best possible and there is no converse bound, outside the abelian case. We will see metabelian groups of derived length 2 and arbitrary nilpotency class.

Lecture 9

6 Polycyclic Groups

Definition 6.1 (Polycyclic Group). We say a group G is **polycyclic** if it has an abelian series $(G_i)_{i=0}^k$ with G_i/G_{i+1} cyclic.

Lemma 6.2 (Polycyclic Closed). Let $H \leq G, N \triangleleft G$. Then

- (i) If G is polycyclic, then H is polycyclic,
- (ii) If G is polycyclic, then G/N polycyclic,
- (iii) If $N, G/N$ are polycyclic, then G is polycyclic.

Proof. Similar to Proposition 5.5.

□

Example 6.3 (F.g. Abelian Polycyclic). Let G finitely generated abelian. Then

$$G \cong \bigoplus_{i=0}^n C_i,$$

where C_i is cyclic. Set

$$G_j := \bigoplus_{i=j}^n C_i,$$

then $G_j/G_{j+1} \cong C_j$.

Proposition 6.4. Every polycyclic group is finitely presented.

Proof. Cyclic groups are f.p. and if $N \triangleleft G$ and $N, G/N$ are f.p. so is G .

□

Remark. This distinguishes polycyclic groups from soluble groups, as there are finitely generated but non-finitely presented soluble groups.

Construction 6.5. Any polycyclic group is f.g. Suppose $s_i \in G_i$ is such that

$$G_i/G_{i+1} = \langle s_i G_{i+1} \rangle,$$

and $G_h = \{e\}$. Then $\forall g \in G \exists n_i \in \mathbb{Z}$ such that

$$g = s_0^{n_0} \cdots s_{h-1}^{n_{h-1}}.$$

We call this expression for g a **polycyclic normal form**.

Warning. Some care should be taken in choosing n_i appropriately for uniqueness.

Proposition 6.6 (Polycyclic Characterisation). G is polycyclic iff it has an abelian series (G_i) such that G_i/G_{i+1} are finitely generated abelian groups.

Proof. One side is trivial, as cyclic groups are finitely generated. Conversely, we have that abelian groups are polycyclic, and by Lemma 6.2(iii), extensions of polycyclic groups are polycyclic. $\square$

Aim. We want to work towards showing that f.g. nilpotent groups are polycyclic.

Notation. For $g_1, \dots, g_k \in G$, $k \geq 3$, we'll write recursively

$$[g_1, \dots, g_k] = [[g_1, \dots, g_{k-1}], g_k].$$

If $N \triangleleft G$, $g, h \in G$, we will write $g \equiv h \pmod N$ for $gh^{-1} \in N$. We will moreover write $\gamma_\bullet$ for $\gamma_\bullet(G)$.

Lemma 6.7. Let $k, m, n \geq 1$, $x \in \gamma_k(G)$, $y \in \gamma_m(G)$, $z \in \gamma_n(G)$. Then

- (i) $[x, yz] \cong [x, y][x, z] \pmod{\gamma_{k+m+n}}$ and $[xy, z] \cong [x, z][y, z] \pmod{\gamma_{k+m+n}}$,
- (ii) $[x, y^{-1}] \cong [x, y]^{-1} \pmod{\gamma_{k+m+n}}$ and $[x^{-1}, y] \cong [x, y]^{-1} \pmod{\gamma_{k+m+n}}$.

Proof. Recall that by Theorem 5.24 we have

$$[\gamma_i, \gamma_j] \leq \gamma_{i+j}.$$

For (i), by Lemma 5.22, we have

$$[x, yz] = [x, z][x, y][[x, y], z],$$

where $[[x, y], z] \in \gamma_{k+m+n}$. Moreover,

$$\begin{aligned} [[x, z], x, y] &\in \gamma_{2k+m+n} \leq \gamma_{k+m+n} \\ \implies [x, z][x, y] &\cong [x, y][x, z] \pmod{\gamma_{k+m+n}}, \end{aligned}$$

and similarly for $[xy, z]$. For (ii), we set $z = y^{-1}$ in (i). $\square$

Proposition 6.8. Let $\langle S \rangle = G$ be any group, and let $T \subseteq \gamma_l(G)$ with

$$\gamma_l(G)/\gamma_{l+1}(G) = \langle t\gamma_{l+1}(G) : t \in T \rangle.$$

Then

$$\gamma_{l+1}(G)/\gamma_{l+2}(G) = \langle [s, t]\gamma_{l+1}(G) : s \in S, t \in T \rangle.$$

Proof. Let $g \in G, h \in \gamma_l(G)$. Then $\exists s_i \in S, t_i \in T, z \in \gamma_{l+1}(G), \delta, \varepsilon \in \{\pm 1\}$ such that

$$\begin{aligned} g &= s_1^{\delta_1} \cdots s_k^{\delta_k}, \\ h &= t_1^{\varepsilon_1} \cdots t_m^{\varepsilon_m} z. \end{aligned}$$

Then

$$\begin{aligned} [g, h] &\cong [g, hz^{-1}][g, z] \\ &\cong [s_1^{\delta_1} \cdots s_k^{\delta_k}, t_1^{\varepsilon_1} \cdots t_m^{\varepsilon_m}] \\ &\cong \prod_{i=j} [s_i, t_j]^{\delta_i \varepsilon_j} \\ &\cong \prod_i [s_i, t_i]^{\delta_i \varepsilon_i} \pmod{\gamma_{l+2}}, \end{aligned}$$

hence $[g, h] \in [S, T]$. □

Corollary 6.9. Let $G = \langle S \rangle$, and

$$T_k = \{[s_1, \dots, s_k] : s_i \in S\},$$

then

$$\gamma_k(G)/\gamma_{k+1}(G) = \langle t\gamma_{k+1}(G) : t \in T_k \rangle.$$

Proof. Inductively apply Proposition 6.8 starting at $T_1 = S$. □

Theorem 6.10. Every finitely generated nilpotent group is polycyclic.

Proof. By Proposition 6.6, suffices to prove

$$\forall i, \gamma_i(G)/\gamma_{i+1}(G),$$

is f.g. Since S is finite, T_k is finite for all k , hence, by Corollary 6.9, we're done. □

7 Growth of Finitely Generated Groups

Remark. A good reference for this Section is [Man11]. See in particular the last Chapter on open problems.

Definition 7.1 (Word Length). Let $G = \langle S \rangle$ be a group with S finite. The **word length** $|g|_S$ of $g \in \langle S \rangle$ is the least n which there exists an expression of the form

$$g = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n},$$

with $s_i \in S, \varepsilon_i \in \{\pm 1\}$.

Definition 7.2 (Word Metric). We define the **word metric** with respect to S as

$$d_S(g, h) := |g \cdot h^{-1}|_S.$$

For $l \in \mathbb{N}$, define the **ball in word length** $B_S(l) := \{g \in G, |g|_S \leq l\}$.

Definition 7.3 (Growth Function). The **growth function** of G w.r.t. S is

$$\begin{aligned} \gamma_G^S : \mathbb{N} &\longrightarrow \mathbb{N} \\ l &\longmapsto |B_S(l)|. \end{aligned}$$

Lecture 10

Question. Are growth functions computable at all? Are they group invariants? What functions can arise as growth functions?

Remark. We have

$$B_S(l) \subseteq B_S(l+1) = \{gx : g \in B_S(l), x \in S \cup S^{-1} \cup \{e\}\}.$$

Thus if $|B_S(l+1)| = |B_S(l)|$, then $B_S(l+m) = B_S(l)$. But $\cup_{k \in \mathbb{N}} B_S(k) = G$, so $B_S(l) = G$ and G is finite. Hence, for any G infinite

$$\forall l \in \mathbb{N} : \gamma_G^S(l+1) \geq \gamma_G^S(l) + 1 \geq l + 2.$$

Remark. Hence the growth function is bounded for finite groups and at least linear for infinite groups.

Example 7.4 (Growth Function of $\mathbb{Z}$). $\gamma_{\mathbb{Z}}^{\{1\}}(l) = 2l + 1$.

Example 7.5 (Growth Function of Products). $G = \langle S \rangle, H = \langle T \rangle$ with $|S|, |T| < \infty$, then $\langle U \rangle = G \times H$, where

$$U = (S \cup S^{-1} \cup \{e_G\}) \times (T \cup T^{-1} \cup \{e_H\}).$$

Note $B_U(l) = B_S(l) \times B_T(l)$. Thus

$$\gamma_{G \times H}^U(l) = \gamma_G^S(l) \cdot \gamma_H^T(l).$$

Example 7.6 (Growth Function of $\mathbb{Z}^d$). By Examples 7.4 and 7.5, $\mathbb{Z}^d$ has generating set U_d such that

$$\gamma_{\mathbb{Z}^d}^{U_d}(l) = (2l + 1)^d.$$

Example 7.7. If $2 \leq |X| < \infty$, define $R_X(l)$ to be the number of reduced words in X of length l . Then

$$R_X(l) = 2|X| \cdot (2|X| - 1)^{l-1},$$

as $x_{i+1} \neq x_i^{-1}$. This gives

$$\gamma_{F(X)}^X(l) = 1 + R_X(1) + \cdots + R_X(l),$$

which grows exponentially with $|X|$.

Example. For any $G = \langle S \rangle, |S| < \infty$, we have the bound

$$\gamma_G^S(l) \leq (\cdots)(|S|),$$

with expression above.

Definition 7.8 (Polynomial Growth). G has **polynomial growth** if $\exists S \subseteq G$ finite s.t. $\exists C > 0$ s.t.

$$\forall l \in \mathbb{N} : \gamma_G^S(l) \leq Cl^C.$$

Definition 7.9 (Exponential Growth). G has **exponential growth** if $\exists S \subseteq G$ finite s.t. $\exists C > 0, \alpha > 1$ s.t.

$$\forall l \in \mathbb{N} : \gamma_G^S(l) \geq C\alpha^l.$$

Definition 7.10 (Intermediate Growth). We say G has **intermediate growth** if it has neither polynomial nor exponential growth.

Lemma 7.11. $H \leq G, H = \langle T \rangle, |T| < \infty$, then $\exists C > 0$ s.t.

$$\forall l \in \mathbb{N} : \gamma_H^T(l) \leq \gamma_G^S(C \cdot l).$$

Proof. Let $C = \max\{|t|_S : t \in T\}$. Then $B_T(l) \subseteq B_S(C \cdot l)$. □

Corollary 7.12. If $G = \langle S \rangle = \langle T \rangle$ with $|S|, |T| < \infty$, then $\exists C > 0$ s.t.

$$\forall l \in \mathbb{N} : \gamma_G^S(l/C)/C \leq \gamma_G^S(l) \leq C\gamma_G^S(C \cdot l). \quad (\star)$$

Proof. Apply Lemma 7.11 in both directions. □

Notation. We'll write $\gamma_G^S \approx \gamma_H^S$ if the bound $(\star)$ holds and write γ_G for the equivalence class of γ_G^S .

Remark. G has polynomial (resp. exponential) growth iff it has polynomial (resp. exponential) growth with respect to every finite generating set S . Note that C, α may still depend on S .

Question 7.13 (Milnor '68). Do there exist f.g. groups of intermediate growth?

Answer. Yes, we will construct one in Section 7.3

Remark. We essentially only have one example of this. We're very far from understanding the structure of groups of intermediate growth.

7.1 Growth of Soluble Groups

Definition 7.14 (Virtually). G is **virtually** $\mathcal{P}$ if $\exists H \leq G$ such that $|G : H| < \infty$ and H has $\mathcal{P}$.

Theorem 7.15 (Milnor-Wolf '68). We have

- (i) If G is finitely generated virtually nilpotent, then G has polynomial growth.
- (ii) If G is finitely generated virtually soluble, not virtually nilpotent, then G has exponential growth.

Remark. See [Mil68, Wol68, Gri13].

Theorem 7.16 (Gromov '81). Every finitely generated group of polynomial growth is virtually nilpotent.

Proof. Beyond this course. Uses tools from Metric Geometry and Non-standard Analysis. See [Gro81, Kle07, ST10, Oza16]. $\square$

Example 7.17. Take $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{GL}(\mathbb{Q})$. Then $G = \langle A, B \rangle$ is soluble of derived length 2 and has exponential growth.

Proposition 7.18. If G is finitely generated and $H \leq_f G$ i.e. $|G : H| < \infty$, then $\gamma_G \approx \gamma_H$.

Proof. Recall that G finitely generated, $H \leq_f G \implies H$ finitely generated. Suppose $G = \langle S \rangle, |S| < \infty$ and T is a transversal to H in G , with $e \in T$.

Claim. $U = H \cap \{t_1 s t_2^{-1} : t_i \in T, s \in S \cup S^{-1}\}$ is a finite generating set for H .

Proof. Take $h \in H$, write

$$h = s_1 \cdots s_n,$$

with $s_i \in S \cup S^{-1}$. Set $t_0 = e, t_{i+1} \in T$ given by

$$H t_i s_{i+1} = H t_{i+1},$$

so that $t_i s_{i+1} t_{i+1}^{-1} \in U$ and

$$h = (t_0 s_1 t_1^{-1}) \cdot (t_1 s_2 t_2^{-1}) \cdots (t_{n-1} s_n e^{-1}).$$

$\square$

This proves $|h|_U \leq |h|_S$, so for every $n \in \mathbb{N}$, we have

$$H \cap B_S(n) \subseteq B_U(n).$$

Hence $\exists t \in T$ such that

$$\frac{1}{|G:H|} |B_S(n)| \leq |Ht \cap B_S(n)|.$$

Let $g \in (Ht) \cap B_S(n)$. Then

$$(Ht \cap B_S(n))g^{-1} \subseteq H \cap B_S(2n) \subseteq B_U(2n).$$

□

Lecture 11

7.2 Relative Growth

Definition 7.19 (Relative Growth). Let $H \leq G$, not necessarily finitely generated. The **relative growth function**

$$\gamma_{H \leq G}^S(n) = |H \cap B_S(n)|.$$

Remark. This depends on S only up to $\approx$.

Remark. If $H = \langle T \rangle$, $|T| < \infty$, then

$$\gamma_H^T(n) \leq \gamma_{H \leq G}^{S \cup T}(n) \approx \gamma_{H \leq G}^S(n).$$

Idea. The difference between relative growth function $\gamma_{H \leq G}^T$ and γ_G^S is related to how "distorted" the subgroup H is inside of G , in terms of metric geometry. The restricted generating set might forbid some significant shortcuts in moving about H .

Proposition 7.20. For $N \triangleleft G, n \in \mathbb{N}$,

$$\gamma_G^S(n) \leq \gamma_{N \leq G}^S(2n) \cdot \gamma_{G/N}^{\bar{S}}(n),$$

where $\bar{S} = \{sN : s \in S\}$.

Proof. Note that if

$$\begin{aligned} \pi : G &\longrightarrow G/N \\ g &\longmapsto gN, \end{aligned}$$

then $\pi(B_S(n)) = B_{\bar{S}}(n)$, and for $g, h \in B_S(n)$, with $\pi(g) = \pi(h)$, then $gh^{-1} \in N \cap B_S(2n)$, so fibres of $\pi|_{B_S(n)}$ have size $\leq |N \cap B_S(2n)|$. □

Theorem 7.21 (Wolf '68). If G is a finitely generated virtually nilpotent group, then G has polynomial growth.

Proof. By Proposition 7.20, we can assume that G is nilpotent of class c . We will induct on c . Consider the derived subgroup $N = [G, G]$, so that $G/N \cong \mathbb{Z}^d \oplus F$, for $|F| < \infty$.

Step 1: set up *nice* generating set. Let $z_i \in G$ be such that $\pi(z_i) \in G/N$ generates the i -th $\mathbb{Z}$ -factor, take $S_F \subseteq G$ finite such that $\langle \pi(S_F) \rangle = F$, and $S_N \subseteq N$ finite such that $\langle S_N \rangle = N$. Note that this can be done as G nilpotent and thus polycyclic, hence N is polycyclic and thus finitely generated. Then

$$S = \{z_1^{\pm 1}, \dots, z_d^{\pm 1}\} \cup \{s^{\pm 1} | s \in S_F\} \cup \{s^{\pm 1} | s \in S_N\},$$

generates G and $|S| < \infty$.

Step 2: We *adapt* S to the lower central series. Recall that by Corollary 6.9 for $k \geq 2$,

$$\gamma_k(G)/\gamma_{k+1}(G) = \langle t\gamma_{k+1}(G) : t \in T_k \rangle,$$

where $T_k = \{[s_1, \dots, s_k] : s_i \in S\}$. Then $T = \bigcup_{k=2}^c T_k$ generated $\gamma_2(G)$. Now take $S' = \{z_1^{\pm 1}, \dots, z_d^{\pm 1}\} \cup \{t^{\pm 1} | t \in T\}$ and $H = \langle S' \rangle \leq G$, so that $N \leq H$ and $\pi(H) = \mathbb{Z}^d$.

Step 3: We have $|G : H| = |F| < \infty$, so by Proposition 7.18, G has polynomial growth iff H does. We have thus reduced to considering free abelian subgroups only.

Step 4: By Proposition 7.20,

$$\gamma_H^S(n) \leq \gamma_{N \leq H}^{S'}(2n) \cdot \gamma_{\mathbb{Z}^d}^{\pi(S')}(n) \leq Cn^d \gamma_{N \leq H}^{S'}(2n).$$

by Example 7.4, so suffices to show $\gamma_{N \leq H}^{S'}$ is polynomial.

Step 5: N is finitely generated nilpotent of class $\leq c-1$, so by induction, N has polynomial growth. Hence suffices to prove that N is not *too distorted* inside G , i.e.

$$\exists C > 0. \forall n \in \mathbb{N}. N \cap B_S(n) \leq B_T(C \cdot n^C).$$

Let $g \in N \cap B_S(n)$, $g = x_1 \cdots x_m$, $m \leq n$, $x_i \in S'$. Write $X_i := \{x_k | x_k \in T_i\} \subseteq T_i$ and note that $\{x_1, \dots, x_m\} = \bigcup_{i=1}^n X_i$. Note that these sets are not disjoint and hence the sum of the $|X_i|$ need not be exactly n .

If $|X_1| = 0$, then $g \in B_T(n)$, so we take $C \geq 1$. If $|X_2| > 0$, then $\pi(g) = 0 \in \mathbb{Z}^d$, without loss of generality, $\exists 1 \leq i < j \leq m$, such that $x_i = z_1, x_j = z_1^{-1}$. So

$$\begin{aligned} g &= \cdots x_{i-1} z_1 x_{i+1} \cdots x_{j-1} z_1^{-1} x_{j+1} \\ &= \cdots x_{i-1} z_1 x_{i+1} \cdots x_{j-2} z_1^{-1} x_{j-1} [x_{j-1}, z_1^{-1}] x_{j+1} \\ &\vdots \\ &= \cdots x_{i-1} z_1^{-1} x_{i+1} [x_{i+1}, z_1^{-1}] \cdots \end{aligned}$$

If $x_i \in T_k$, $k \leq c-1$, then $[x_i, z_1^{-1}] \in T_{k+1}$ and we add it to X_{k+1} and if $x_i \in T_c$, then $[x_1, z_1^{-1}] = e$. We now replace $|X_1|$ by $|X'_1| \leq |X_1| - 2$, $|X_2|$ by $|X'_2| \leq |X_2| + |X_1|$ and $|X_c|$ by $|X'_c| \leq |X_c| + |X_{c-1}|$. This addresses index overlaps at X_1 step. Writing $|X_1| = 2l$, we iterate the process l steps, terminating with $X_1^{(l)} = \emptyset$. Our final word length will be

$$|\bigcup_{i=2}^c X_i^{(l)}| \leq Cn^C.$$

□

Remark. Let us remark that nilpotency is crucial here, since it kills the commutators after c steps. This avoids exponential build-ups.

Lecture 12

We now turn to non-nilpotent groups. Let $G = \langle S \rangle$, $|S| < \infty$ in what follows.

Lemma 7.22. For $N \triangleleft G$, if G/N has exponential growth, then so does G .

Proof. We have

$$\begin{aligned} \pi : G &\longrightarrow G/N \\ g &\longmapsto g \cdot N \end{aligned}$$

and $\pi(B_S(n)) = B_{\pi(S)}(n)$, thus $\gamma_{G/N}^{\pi(S)}(n) \leq \gamma_G^S(n)$. $\square$

Theorem 7.23 (Milnor '68). Let G be a finitely generated (virtually) soluble group. If G does not have exponential growth, then G is (virtually) polycyclic.

Proof. Induction on the derived length d of G . For $d = 1$, by Example 5.2, G is finitely generated abelian.

For $d \geq 2$, assume that G is not abelian. Set $N = G^{(d-1)}$ so that by Proposition 5.11, $N \triangleleft G$, and $[N, N] = G^{(d)} = \{e\}$, so N is abelian. Moreover, G/N has derived length $(d-1)$ and by hypothesis and Lemma 7.22, we know that it does not have exponential growth. Hence G/N is polycyclic.

It now suffices to show that N is finitely generated, by Lemma 6.2(iii) and Example 6.3. Proceed in steps:

Claim. $\forall g \in G, n \in N$, $\langle g^i n g^{-i} : i \in \mathbb{Z} \rangle$ is finitely generated.

Proof. For $\vec{\varepsilon} \in \{0, 1\}^{k+1}$, let

$$w(\vec{\varepsilon}) := n^{\varepsilon_0} g n^{\varepsilon_1} g \cdots n^{\varepsilon_k} g \in B_S((k+1)(a+b))$$

If all $w(\vec{\varepsilon})$ are distinct in G , then

$$\gamma_G^S((k+1)(a+b)) \geq 2^{k+1},$$

and if this holds for all $k \in \mathbb{N}$, then for $l \in \mathbb{N}$, if $k(a+b) \leq l \leq (k+1)(a+b)$ then

$$\gamma_G^S(l) \geq 2^k \geq \frac{1}{2} 2^{k+1} \geq \frac{1}{2} (2^{-(a+b)})^l, \quad (\dagger)$$

is exponential. Since our group does not have exponential growth, we must have $\exists k \in \mathbb{N}, \vec{\varepsilon}, \vec{\delta} \in \{0, 1\}^{k+1}$ such that $w(\vec{\varepsilon}) = w(\vec{\delta})$.

Choosing k minimal, can assume $\varepsilon_k \neq \delta_k$, without loss of generality $\varepsilon_k = 1, \delta_k = 0$.

Then $g^k n g^{-k} = n^{\delta_0 - \varepsilon_0} (g n g^{-1})^{\delta_1 - \varepsilon_1} \cdots (g^{k-1} n g^{-(k-1)})^{\delta_{k-1} - \varepsilon_{k-1}}$ and since N is abelian, $g^i n g^{-i} \in N$. Thus $g^k n g^{-k} \in \langle g^i n g^{-i} : 0 \leq i \leq k-1 \rangle =: H_{k-1}$. This gives $H_k = H_{k-1}$. Now $g^{k+1} n g^{-(k+1)} = g(g^k n g^{-k})g^{-1} \in g H_{k-1} g^{-1} \subseteq H_k = H_{k-1}$. This gives $H_{k+1} = H_{k-1}$.

By induction, $H_i = H_{k-1}$, for all $i \geq k' - 1$, so

$$\langle g^i n g^{-i} : i \geq 0 \rangle = \bigcup_{i=k-1}^{\infty} H_i = H_{k-1},$$

and thus is finitely generated. Similarly, $\langle g^i n g^{-i} : i \leq 0 \rangle$ is also finitely generated, which gives that $\langle g^i n g^{-i} : i \in \mathbb{Z} \rangle$. $\square$

This covers the case where N is cyclic and normally generated by one element.

Claim. $\exists A \subseteq N$ finite such that $N = \langle \langle A \rangle \rangle^G$.

Proof. Let $R \subseteq F(S)$, not necessarily finite, such that $G = \langle S | R \rangle$. Let $\tilde{A} \subseteq N$ with $N = \langle \langle \tilde{A} \rangle \rangle^G$, again not necessarily finite. For $a \in \tilde{A}$, let $w_a \in F(S)$ with $w_a =_G a$. Then $G = \langle S | R \cup \{w_a : a \in \tilde{A}\} \rangle$. But G/N is polycyclic, hence finitely presented by Prop. 6.4. By Proposition 2.29, $\exists R_0 \subseteq R$ finite, $A \subseteq \tilde{A}$ finite,

$$G/N = \langle S | R_0 \cup \{w_a : a \in A\} \rangle.$$

Thus in $F(S)$, $\forall a \in \tilde{A}$, we have

$$w_a \in \langle \langle R_0 \cup \{w_b : b \in A\} \rangle \rangle^{F(S)},$$

so in $G = F(S) / \langle \langle R \rangle \rangle^{F(S)}$, have

$$\forall a \in \tilde{A}, a = w_a \in \langle \langle \{w_b : b \in A\} \rangle \rangle^G = \langle \langle A \rangle \rangle^G.$$

Hence $N = \langle \langle A \rangle \rangle^G$. □

Claim. N is finitely generated.

Proof. G/N is polycyclic, so by Remark 6.5,

$$\exists s_0, \dots, s_{h-1} \in G. \forall g \exists m_i \in \mathbb{Z}, n \in N. g = s_0^{m_0} \cdots s_{h-1}^{m_{h-1}} n$$

Then $N = \langle s_0^{m_0} \cdots s_{h-1}^{m_{h-1}} a s_{h-1}^{-m_{h-1}} \cdots s_0^{-m_0} : m_i \in \mathbb{Z}, a \in A \rangle$. Consider $N_n = \langle A \rangle$, finitely generated, given by

$$N_i := \langle s_i^{m_i} \cdots s_{h-1}^{m_{h-1}} a s_{h-1}^{-m_{h-1}} \cdots s_i^{-m_i} : m_j \in \mathbb{Z}, a \in A \rangle,$$

with $N_0 = N$. We now claim that for $1 \leq i \leq h$, N_i finitely generated, then N_{i-1} is finitely generated. If $N_i = \langle X_i \rangle$, $|X_i| < \infty$, then $N_{i-1} = \langle s_{i-1}^m x s_{i-1}^{-m} : m \in \mathbb{Z}, x \in X_i \rangle$. But for each $x \in X$, $\langle s_{i-1}^m x s_{i-1}^{-m} : m \in \mathbb{Z} \rangle$ is finitely generated by some Y_x , as previously shown. Then $N_{i-1} = \langle \bigcup_{x \in X_i} Y_x \rangle$ is finitely generated. This concludes the proof. □

Remark. Note that we explicitly use sub-exponential growth only in Equation (†). □

Theorem 7.24 (Tits '72). If k is a field, $G \leq \text{GL}_d(k)$ finitely generated. Either G is virtually soluble or G has a non-abelian free subgroup.

Proof. Beyond scope of course, see [Tit72]. □

Corollary 7.25. If $G \leq \text{GL}_d(\mathbb{K})$ is finitely generated, then G does not have intermediate growth.

Proof. Use Tits' Theorem to reduce by cases, Milnor-Wolf and Example 7.7. □

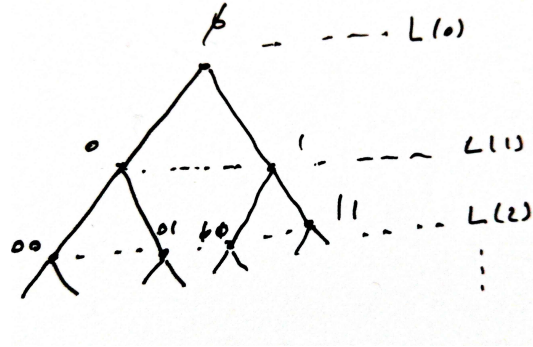


Figure 1: The rooted binary tree and its level sets

Theorem 7.26 (Wolf '68 [Wol68]). G finitely generated polycyclic. If G does not have exponential growth then G is virtually nilpotent.

Sketch of Proof. It uses Auslander-Swan '68 result [Aus67, Swa67] about structure of polycyclic groups: G is isomorphic to a subgroup of $\mathrm{GL}_d(\mathbb{Z})$ for some $d \geq 2$.

It also uses that $G \leq \mathrm{GL}_d(\mathbb{C})$ soluble implies $\exists H \leq_f G, g \in \mathrm{GL}_d(\mathbb{C})$ such that gHg^{-1} is upper triangular.

Note that $H \leq \mathrm{GL}_d(\mathbb{Z})$ so $\exists N = N(d) \in \mathbb{N}$ such that $\forall h \in H$, every e-value $\lambda \in \mathbb{C}$ of h with $|\lambda| = 1$ satisfies $\lambda^N = 1$. Then either $\exists K \leq_f H$ such that gKg^{-1} is upper triangular, or $\exists h \in H$ such that h has an e-value $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. If it was the second case, we could use it to prove exponential growth. See Example 7.17 and other 3x3 matrix exercise on ES#2. $\square$

7.3 The Grigorchuk Group

Aim. Construct a group that satisfies both Question 5.14 and Question 7.13.

Definition 7.27 (Rooted Binary Tree, Figure 1). The **rooted binary tree** is the graph $T_2 = (V, E)$, where

$$V = \{0, 1\}^* = \coprod_{n \in \mathbb{N}} \{0, 1\}^n$$

$$E = \{(v, v0), (v, v1) : v \in V\}.$$

Every graph has an associated automorphism group.

Notation. We denote:

$$\mathrm{Aut}(T_2) = \{g \in \mathrm{Symm}(V) : \forall v, w \in V, (v, w) \in E \iff (g(v), g(w)) \in E\}.$$

Note that $g \in \mathrm{Aut}(T_2)$ satisfies $g(\emptyset) = \emptyset$.

Remark. Let $L(n) = \{x_1, \dots, x_n : x_i \in \{0, 1\} \subseteq V\}$ be the n -th **level** of T_2 . Then $L(n)$ consists of vertices at distance exactly n from $\emptyset$, with respect to path metric. Hence $\forall g \in \mathrm{Aut}(T_2), g(L(n)) = L(n)$.

Definition 7.28 (Level Stabiliser). We define the n -th **level stabiliser** as the kernel of the action of $\text{Aut}(T_2)$ on $L(n)$

$$\text{Stab}(n) = \{g \in \text{Aut}(T_2) : \forall v \in L(n), g(v) = v\} \triangleleft \text{Aut}(T_2).$$

Remark. We have $\text{Stab}(n+1) \leq \text{Stab}(n)$.

Lemma 7.29. There is an isomorphism

$$\begin{aligned} \varphi : \text{Aut}(T_2) \times \text{Aut}(T_2) &\longrightarrow \text{Stab}(1) \\ \varphi(g, h)(0v) &\longmapsto 0g(v) \\ \varphi(g, h)(1v) &\longmapsto 1h(v). \end{aligned}$$

Proof. Certainly $\varphi(g, h) \in \text{Stab}(1)$. Consider its action on vertices

$$\begin{aligned} (\varphi(g_1, h_1)\varphi(g_2, h_2))(0v) &= 0(g_1g_2)(v) \\ &= \varphi(g_1g_2, h_1h_2)(0v), \end{aligned}$$

and likewise for $1v$, so φ is a homomorphism. If $\varphi(g, h) = \text{id}_{T_2}$, then $\forall v \in V$,

$$\begin{aligned} 0g(v) &= 0v \\ 1h(v) &= 1v \\ g(v) &= h(v) = v \\ g &= h = \text{id}_{T_2} \end{aligned}$$

For $k \in \text{Stab}(1)$, define

$$g, h : V \longrightarrow V,$$

by $k(0v) = 0g(v)$, $k(1v) = 1h(v)$. Then $g, h \in \text{Aut}(T_2)$, since $k \in \text{Aut}(T_2)$ and $k = \varphi(g, h)$. $\square$

Corollary 7.30. $\varphi(\text{Stab}(n) \times \text{Stab}(n)) = \text{Stab}(n+1)$

Remark. We will often suppress φ , writing $(g, h) \in \text{Stab}(1)$.

Definition 7.31. We define $a \in \text{Aut}(T_2)$ by

$$\begin{aligned} a(0v) &= 1v \\ a(1v) &= 0v, \end{aligned}$$

for all $v \in V$.

Remark. Note that $\langle a \rangle \cong C_2$.

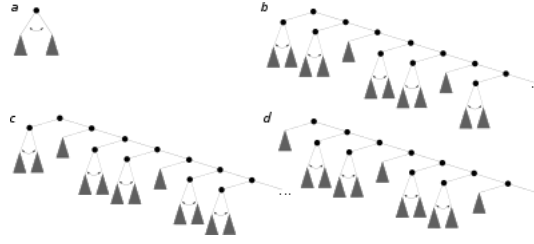


Figure 2: The action of the Grigorchuk group's generators.

Proposition 7.32. We have

$$\text{Aut}(T_2) = \text{Stab}(1) \rtimes \langle a \rangle,$$

where $a(g, h)a^{-1} = (h, g)$.

Proof. We have

- (i) $\text{Stab}(i) \triangleleft \text{Aut}(T_2)$, as in Definition 7.28
- (ii) $a(0) = 1$, so $a \notin \text{Stab}(1)$ and $\langle a \rangle \cap \text{Stab}(1) = \{e\}$,
- (iii) $\forall g \in \text{Aut}(T_2)$, either $g(0) = 0, g(1) = 1$, so $g \in \text{Stab}(1)$ or $g(0) = 1, g(1) = 0$ so $(ga) \in \text{Stab}(1)$ and $g = (ga)a$. Finally, $\forall v \in V$,

$$\begin{aligned} (a(g, h)a^{-1})(0v) &= (a(g, h))(1v) \\ &= a(1h(v)) \\ &= 0h(v) \\ &= (h, g)(0v), \end{aligned}$$

and likewise $(a(g, h)a^{-1})(1v) = (h, g)(1v)$.

□

Remark. $\text{Aut}(T_2)$ is not our target group, as it is not even finitely generated.

Definition 7.33 (Grigorchuk Group). The **Grigorchuk Group** is

$$\mathbb{G} = \langle a, b, c, d \rangle,$$

where $b, c, d \in \text{Stab}(1)$ are defined by

$$\begin{aligned} b &= (a, c) \\ c &= (a, d) \\ d &= (c, b) \end{aligned}$$

Remark. This is actually well defined, as we're describing what each element does in terms of what other elements do one level down. See Figure 2.

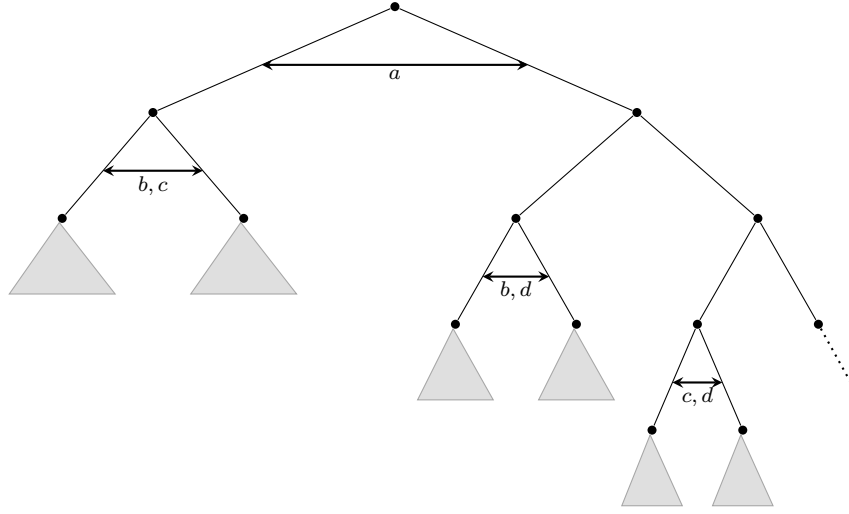


Figure 3: The action of the generators of $\mathbb{G}$: each arrow swaps the two subtrees on either side of it.

Aim. We want to prove that

- (i) $\mathbb{G}$ is an infinite finitely generated torsion group.
- (ii) $\mathbb{G}$ is a finitely generated group of intermediate growth.

Remark. Historically not the first found to answer Burnside problem, but it was the first to answer Milnor's problem.

Question. Does Milnor imply Burnside?

Lemma 7.34 (Grigorchuk Relations). In $\mathbb{G}$, we have $b^2 = c^2 = d^2 = bcd = e$.

Proof. We have $g \in \text{Aut}(T_2)$, $g = e \iff \forall n \in \mathbb{N}, g \in \text{Stab}(n)$. We prove this by induction on n . Since $b, c, d \in \text{Stab}(1)$, we have $b^2, c^2, d^2, bcd \in \text{Stab}(1)$. Suppose $b^2, c^2, d^2, bcd \in \text{Stab}(n)$.

$$b^2 = (a, c) \cdot (a, c) = (a^2, c^2) = (e, c^2) \in \text{Stab}(n+1)$$

$$c^2 = (a^2, d^2) = (e, d^2) \in \text{Stab}(n+1)$$

$$d^2 = (e, b^2) \in \text{Stab}(n+1)$$

$$bcd = (a, c) \cdot (a, d) \cdot (e, b) = (a^2, cdb) = (e, b^{-1}(bcd)b) \in \text{Stab}(n+1).$$

□

Corollary 7.35. $\langle b, c, d \rangle \cong V_4$.

Corollary 7.36. There is a surjective homomorphism

$$\rho : C_2 * V_4 \cong \langle a \rangle * \langle b, c, d \rangle \longrightarrow \mathbb{G}$$

Corollary 7.37. Every element of $\mathbb{G}$ admits a reduced expression

$$a^\delta x_1 a x_2 \cdots a x_n a^\varepsilon,$$

$\delta, \varepsilon \in \{0, 1\}, x_i \in \{b, c, d\}.$

Remark 7.38. The homomorphism is not injective:

$$adad = a(e, b)a^{-1}(e, b) = (b, e)(e, b) = (b, b),$$

and hence $(ad)^4 = (b^2, b^2) = e$ a non-trivial reduced expression for the identity.

Remark. For $g \in \mathbb{G}$ we have $g \in \text{Stab}_{\mathbb{G}}(1) := \mathbb{G} \cap \text{Stab}(1)$ if and only if some, or equivalently any, reduced expression for g has an even number of as .

Thus $\text{Stab}_{\mathbb{G}}(1) = \langle b, c, d, aba, aca, ada \rangle.$

Notation. Define

$$\begin{aligned} \varphi_0, \varphi_1 : \text{Stab}_{\mathbb{G}}(1) &\longrightarrow \text{Aut}(T_2) \\ \varphi_0(g, h) &\longmapsto g \\ \varphi_1(g, h) &\longmapsto h. \end{aligned}$$

Lemma 7.39. $\text{im}(\varphi_0) = \text{im}(\varphi_1) = \mathbb{G}.$

Proof. We have all generators in each image:

x	$\varphi_0(x)$	$\varphi_1(x)$
b	$\underline{a}$	$\underline{c}$
c	a	$\underline{d}$
d	e	$\underline{d}$
aba	$\underline{c}$	$\underline{a}$
aca	$\underline{d}$	a
ada	$\underline{b}$	e

□

Theorem 7.40. $|\mathbb{G}| = \infty.$

Proof. $\mathbb{G}$ has a proper subgroup, $\text{Stab}_{\mathbb{G}}$ which admits a surjective homomorphism onto $\mathbb{G}$, i.e. $(\varphi_0, \varphi_1).$ □

Theorem 7.41. $\forall g \in \mathbb{G}, \exists n \in \mathbb{N}$ such that $g^{2^n} = e.$

Proof. Induction on $l = |g|_S$, the length of the shortest reduced expression. For $l = 1$, $g \in S$ has order 2 by Lemma 7.34. For $l = 2$, g is conjugate to ab, ac or ad and thus it has their same order. By Remark 7.38, $(ad)^4 = e$. Moreover,

$$(ac)^4 = aca \cdot c = (d, a) \cdot (a, d) = (da, ad),$$

and hence $(ad)^8 = e$, since $da = a(ad)a^{-1}$. Finally $(ab)^2 = aba \cdot b = (ca, ac)$ and thus $(ab)^{16} = e$.

For $l \geq 3$ odd, either $g = awa$, which is conjugate to w and thus $|w|_S < l$, or $g = xwy$ is conjugate to $w(yx)$, and thus $x, y \in \{b, c, d\}$, $yx \in \{e, b, c, d\}$, which gives $|w(yx)|_S < l$.

For $l \geq 4$ even, g is conjugate to $ax_1ax_2 \cdots ax_m = w$, with $l = 2m, x_i \in \{b, c, d\}$. We have two cases:

(1) $m = 2k$ is even, then

$$w = (ax_1a)x_2(ax_3a)x_4 \cdots (ax_{2k-1}a)x_{2k}.$$

Hence $x_i = (a^{\varepsilon_i}, y_i), \varepsilon_i \in \{0, 1\}, y_i \in \{e, b, c, d\}$ and $ax_ia = (y_i, e^{\varepsilon_i})$. Thus

$$\begin{aligned} g &= (g_0, g_1) \\ g_0 &= y_1 a^{\varepsilon_2} y_3 a^{\varepsilon_4} \cdots y_{2k-1} a^{\varepsilon_{2k}} \\ g_1 &= a^{\varepsilon_1} y_2 a^{\varepsilon_3} y_4 \cdots a^{\varepsilon_{2k-1}} y_{2k}. \end{aligned}$$

Then $|g_0|_S, |g_1|_S \leq 2 = m = l/2$, and $g^{2^m} = (g_0^{2^m}, g_1^{2^m})$ by induction.

(2) $m = 2k - 1$ odd, then

$$w = (ax_1a)x_2(ax_3a)x_4 \cdots ax_{2k-1},$$

and

$$\begin{aligned} w^2 &= (ax_1a)x_2 \cdots x_{2k-2}(ax_{2k-1}a) \\ &\quad \cdot x_1(ax_2a)x_3 \cdots (ax_{2k-2}a)x_{2k-1}, \end{aligned}$$

so $g^2 = (g_0, g_1)$ where

$$g_0 = (y_1 a^{\varepsilon_2} \cdots a^{\varepsilon_{2k-2}} y_{2k-1}) \cdot (a^{\varepsilon_1} y_2 \cdots y_{2k-2} a^{\varepsilon_{2k-1}}) = h_1 \cdot h_2.$$

- (i) $\exists i. x_i = d = (e, b)$. Then $\varepsilon_i = 0$ and $|g_0|, |g_1| \leq l - 1$, thus we're done by induction.
- (ii) $\exists i. x_i = c = (a, d)$. If some $\varepsilon_j = 0$, $|g_0|, |g_1| \leq l - 1$. If not g_0, g_1 are as in case 2(i).
- (iii) $\forall i. x_i = b$. Then $g \in \langle ab \rangle$, so $\text{ord}(g) | 16$, by case $l = 2$.

□

Remark. This *push it down* behaviour should allow for a co-inductive description of the Grigorchuk group. For example, for $\emptyset \neq g \in T_2$, we may write b as

$$\text{hd}(\text{apply}_b(g)) = (\text{hd}(\text{apply}_a(g)), \text{hd}(\text{apply}_c(g))).$$

Could other groups of intermediate growth be obtained from descriptions of such sorts?

Corollary 7.42. $\mathbb{G}$ is an answer to Burnside's Question 5.14.

Proof. Immediate from Theorem 7.40 and Theorem 7.41. $\square$

Lecture 15

7.4 Growth of Grigorchuk Group

Aim. Prove $\mathbb{G}$ does not have exponential growth.

Proposition 7.43. $\exists C > 0. \forall g \in \text{Stab}_{\mathbb{G}}(3),$

$$\sum_{\varepsilon_i \in \{0,1\}} |g_{\varepsilon_1 \varepsilon_2 \varepsilon_3}|_S \leq \frac{15}{16} |g|_S + C,$$

where $g = (g_0, g_1) = (g_{00}, g_{01}, g_{10}, g_{11}) = (g_{000}, g_{001}, \dots, g_{111})$.

Proof. Suppose $|g|_S \geq C_0$, for some C_0 sufficiently large. Let w be a reduced expression for g of length $|g|_S$. Write

$$w = x_0^\delta (ax_1 a) x_2 \cdots x_{2n} (ax_{2n+1} a) x_{2n+1}^\varepsilon,$$

where $x_i = (a^{\varepsilon_i}, y_i) \in \{b, c, d\}, y_i \in \{b, c, d\}, \varepsilon, \delta \in \{0, 1\}$. Let $w_0 \in \langle a \rangle * \langle b, c, d \rangle$ be the (unique) reduced word presenting

$$a^{\delta \varepsilon_0} y_1 a^{\varepsilon_2} \cdots a^{\varepsilon_{2n}} y_{2n+1} a^{\varepsilon \varepsilon_{2n+2}} \in \langle a \rangle * \langle b, c, d \rangle,$$

so that $w_0 =_{\mathbb{G}} g_0$. Similarly, let w_1 be the reduced representation for

$$y_0^\delta a^{\varepsilon_1} y_2 \cdots y_{2n+2}^\varepsilon,$$

so $w_1 =_{\mathbb{G}} g_1$.

We note the following two facts:

Lemma 7.44. We have in following

$$|g_0|_S + |g_1|_S \leq |w_0| + |w_1| \leq |g|_S + 1.$$

Lemma 7.45. We have

(i) If w contains $\geq N$ ds , then

$$|w_0| + |w_1| \leq |w| - N + 1.$$

(ii) If w contains $\geq N$ cs (resp. bs), then either we have cancellations

$$|w_0| + |w_1| \leq |w| - \frac{N}{2} + 1,$$

or w_0 and w_1 contain, in total, $\geq \frac{N}{2}$ ds (resp. cs).

Since $g \in \text{Stab}_{\mathbb{G}}(3)$, we get

$$w \rightsquigarrow w_0, w_1 \rightsquigarrow w_{00}, w_{01}, w_{10}, w_{11} \rightsquigarrow w_{000}, w_{001}, \dots, w_{111}.$$

By Lemma 7.44, we have

$$\begin{aligned} & |g_{000}|_S + \dots + |g_{111}|_S \\ & \leq |w_{000}|_S + \dots + |w_{111}|_S \\ & \leq |w_{00}|_S + \dots + |w_{11}|_S + 4 \\ & \leq |w_0|_S + \dots + |w_1|_S + 6 \\ & \leq |w|_S + 7. \end{aligned} \tag{*}$$

Proceed by cases:

(i) w contains $\geq 1 \frac{1}{16}|w|$ ds . By Lemma 7.45(i)

$$(*) \leq |w_0| + |w_1| + 6 \leq \frac{15}{16}|w| + 7.$$

(ii) w contains $\geq \frac{1}{8}|w|$ cs . By Lemma 7.45(ii), either cancellation

$$(*) \leq |w_0| + |w_1| + 6 \leq \frac{15}{16}|w| + 7,$$

or obtain $\geq \frac{1}{16}|w|$ ds between w_0 and w_1 , as in case (i),

$$\begin{aligned} (*) & \leq |w_{00}|_s + \dots + |w_{11}|_s + 4 \\ & \leq |w_0|_s + \dots + |w_1|_s - \frac{1}{16}|w| + 6 \\ & \leq \frac{15}{16}|w|_s + 7. \end{aligned}$$

(iii) w contains $\geq \frac{1}{2}(|w| - 1) - \frac{1}{8}|w| - \frac{1}{16}|w| \geq \frac{1}{4}|w|$ bs , for C_0 sufficiently large, we either have cancellation

$$\begin{aligned} (*) & \leq |w_0| + |w_1| + 6 \\ & \leq \frac{7}{8}|w| + 7, \end{aligned}$$

or obtain $\geq \frac{1}{8}|w|$ cs between w_0 and w_1 , as in case (ii), we have

$$(*) \leq \frac{15}{16}|w| + 7.$$

□

Lemma 7.46 (Moderation of 7.20). Let $N \triangleleft_f G$. Then $\exists M \in \mathbb{N}. \forall n \in \mathbb{N}$,

$$\gamma_G^S(n) \leq |G/N| \cdot \gamma_{N \leq G}^S(n + M).$$

Proof. T finite transversal to N in G . Let $M \in \mathbb{N}$ be such that $T \subseteq B_S(M)$. For $g \in G$, $\exists! t(g) \in T$ such that $gN = t(g)N$. Then

$$\begin{aligned} f : B_S(n) & \longrightarrow T \times (N \cap B_S(n + M)) \\ g & \longmapsto (t(g), t(g)^{-1}g), \end{aligned}$$

is injective. □

Every group exponential growth has an exponential growth rate:

Lemma 7.47 (ES#3). If G has exponential growth, then $\exists \lambda > 1$ such that

$$(\gamma_G^S(n))^{1/n} \rightarrow \lambda,$$

as $n \rightarrow \infty$.

Theorem 7.48. $\mathbb{G}$ does not have exponential growth.

Proof. If not, let $\lambda > 1$ be as in Lemma 7.47, and let $\varepsilon > 0$. Then $\exists C_0(\varepsilon) > 0$ such that $\forall n \in \mathbb{N}$

$$\frac{1}{C_0}(\lambda - \varepsilon)^n \leq \gamma_{\mathbb{G}}^S(n) \leq C_0(\lambda + \varepsilon)^n.$$

By Proposition 7.43,

$$\begin{aligned} \gamma_{\text{Stab}_{\mathbb{G}}(3)}^S(n) &\leq \sum \gamma_{\mathbb{G}}^S(n_1) \cdots \gamma_{\mathbb{G}}^S(n_8) \\ &\leq \sum C_0^8 (\lambda + \varepsilon)^{\sum n_i} \\ &\leq C_1 (\lambda + \varepsilon)^{\frac{15}{16}n + C'} n^8, \end{aligned} \tag{†}$$

where the sum is over

$$\sum n_i \leq \frac{15}{16}n + C.$$

But by Lemma 7.46, we have lower bound

$$(†) \cdot |G : \text{Stab}_{\mathbb{G}}(3)| \geq \gamma_{\mathbb{G}}^S(n - M) \geq \frac{1}{C_0}(\lambda - \varepsilon)^{n - M},$$

and since this holds for all $n \in \mathbb{N}$, we can pass to the limit and get

$$(\lambda + \varepsilon)^{\frac{15}{16}} \geq (\lambda - \varepsilon),$$

and since it holds for all $\varepsilon > 0$,

$$\lambda^{\frac{15}{16}} \geq \lambda > 1,$$

contradiction. □

Lecture 16

Aim. Prove $\mathbb{G}$ does not have polynomial growth.

Proposition 7.49 (ES#3). $\exists K \leq_f \mathbb{G}$ such that $\phi(K \times K) \leq_f K$.

Theorem 7.50. $\mathbb{G}$ does not have polynomial growth.

Proof. By Proposition 7.11,

$$\gamma_{\mathbb{G}} \approx \gamma_K \approx \gamma_{K \times K} \approx \gamma_{\mathbb{G} \times \mathbb{G}} \approx \gamma_{\mathbb{G}}^2,$$

since $K \times K \leq_f \mathbb{G} \times \mathbb{G}$. Thus $\exists C > 1$ such that $\forall n \in \mathbb{N}$

$$C \times \gamma_{\mathbb{G}}^S(C \cdot n) \geq \gamma_{\mathbb{G}}^S(n).$$

By induction on m ,

$$C^m \cdot \gamma_{\mathbb{G}}^S(C^m) \geq \gamma_{\mathbb{G}}^S(1)^{2^m} \geq 4^{2^m}.$$

Thus taking $C^m \leq n \leq C^{m+1}$ we have

$$\gamma_{\mathbb{G}}^S(n) \geq \exp\left(\frac{\log(4)}{2} n^{\frac{\log(2)}{\log(C)}}\right).$$

□

Theorem 7.51 (Bartholdi '98, Erschler+Zhang '18).

$$\gamma_{\mathcal{G}} \approx \exp(n^\alpha) \quad \alpha = 0.7674\dots$$

Proof. A lower bound was found by Bartholdi in [Bar98], whereas the exact growth rate is a recent result by Erschler and Zang [EZ19]. □

Question. (Open) Is there a group of slower intermediate growth than $\mathbb{G}$?

Remark. It has been proven not to be possible for large classes of groups.

8 Amenable Groups

In the following, G will be countable. There is a generalised notion of being amenable for uncountable groups, but it is of topological nature.

Notation. For $A, F \subseteq G$, we write

$$A \cdot F := \{a \cdot f : a \in A, f \in F\}.$$

Definition 8.1 ((A, ε) -Følner Set). $A \subseteq G$ finite, $\varepsilon > 0$, $F \subseteq G$ finite is an (A, ε) -Følner set if

$$|A \cdot F \cup F| \leq (1 + \varepsilon)|F|.$$

Definition 8.2 (Amenable Groups). G is **amenable** if for all (A, ε) , G has an (A, ε) -Følner set.

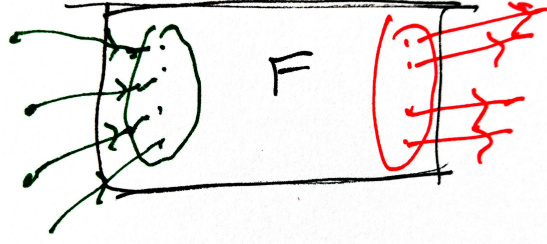
Example 8.3 (Finite Amenable). G finite, suffices to take $F = G$ to obtain all the (A, ε) -Følner sets.

Example 8.4 ($\mathbb{Z}$ Amenable). Let $G = \mathbb{Z}$, $F = \mathbb{Z} \cap [-m, m]$ and set

$$k := \max\{|a| : a \in H\}.$$

Then $A + F \subseteq \mathbb{Z} \cap [-(m+k), (m+k)]$, so

$$|A + F| \leq |F| + 2k,$$

Figure 4: The subsets O_g, I_g of F .

so choose $m > \frac{k}{2\varepsilon}$. Thus $\mathbb{Z}$ is amenable.

Theorem 8.5 (Amenable Sofic). If G is amenable, then G is sofic.

Proof. $A \subseteq G$ finite, $\varepsilon > 0$, $F \subseteq G$ an $(A, \frac{\varepsilon}{3})$ -Følner set. We should to define $\varphi(g) \in \text{Symm}(F)$ for $g \in A$. Since the action

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto g \cdot x, \end{aligned}$$

is a bijection,

$$|O_g| = |\{f \in F : g \cdot f \notin F\}| = |\{f \in F : g^{-1} \cdot f \notin F\}| = |I_g|,$$

as depicted in Figure 4. Set

$$\varphi(g)(\delta) = \begin{cases} g \cdot \delta & \text{if } \delta \notin O_g \\ \Psi_g(\delta) & \text{if } \delta \in O_g \end{cases}$$

where Ψ is any bijection between O_g and I_g . Certainly, $\phi(g) \in \text{Symm}(F)$. We now claim that $\varphi : A \rightarrow \text{Symm}(F)$ is an (A, ε) -representation, i.e.

- (i) For $g, h \in A$, if $gh \in A$, then $d_F(\phi(gh), \phi(g)\phi(h)) < \varepsilon$,
- (ii) For $e \neq g \in A$, $d_F(\phi(g), \text{id}_F) < \varepsilon$.

Note that

$$\forall g \in A : |O_g| \leq \frac{\varepsilon}{3} |F|.$$

For (i), if $\delta \in F \setminus (O_h \cup h^{-1}O_g)$, then $h\delta, gh\delta \in F$, and thus

$$\begin{aligned} (\phi(g)\phi(h))(\delta) &= \phi(g)(h\delta) \\ &= gh\delta \\ &= \phi(gh)(\delta), \end{aligned}$$

and

$$|F \setminus (O_h \cup h^{-1}O_g)| \geq \left(1 - \frac{2\varepsilon}{3}\right) |F|.$$

For (ii), if $f \in F \setminus O_g$, then $\phi(g)(\delta) = g \cdot \delta \neq \delta$ and

$$|F \setminus O_g| \geq \left(1 - \frac{\varepsilon}{3}\right) |F|.$$

□

Remark. Countable assumption might be necessary.

Question. Why is amenability an interesting condition?

Theorem 8.6. If G is f.g. non-amenable, then G has exponential growth.

Proof. Let $A \subseteq G$ be finite, $\varepsilon > 0$ be such that $\neg \exists(A, \varepsilon)$ -Følner set. In particular, $\forall n \in \mathbb{N}, B_A(n)$ is not an (A, ε) -Følner set, hence

$$|A \cdot B_A(n) \cup B_A(n)| \geq (1 + \varepsilon)|B_A(n)|,$$

but

$$A \cdot B_A(n) \cup B_A(n) \subseteq B_A(n+1),$$

hence, letting $H = \langle A \rangle \leq G$,

$$\gamma_H^A(n+1) \geq (1 + \varepsilon)\gamma_H^A(n),$$

hence $\forall n, \gamma_H^A(n) \geq (1 + \varepsilon)^{n-1}$, so H has exponential growth. By Lemma 7.11, G has exponential growth. $\square$

Corollary 8.7. Every finitely generated nilpotent group is amenable.

Corollary 8.8. $\mathbb{G}$ is amenable, and thus sofic.

Theorem 8.9 (Amenable Closed). For $H \leq G, N \triangleleft G$, we have

- (i) If G is amenable, then H is amenable,
- (ii) If G is amenable, then G/N is amenable,
- (iii) If $G/N, N$ is amenable, then G is amenable,
- (iv) Suppose $G = \bigcup_{i \in I} G_i$ for $\{G_i : i \in I\}$ a directed system of groups. If all G_i are amenable, so is G .

Question (Open). Do (i) and (ii) also hold for sofic groups?

Corollary 8.10 (Amenable Characterisation). A countable group G is amenable if and only if every finitely generated subgroup of G is amenable.

Proof. One direction is given by Theorem 8.9(i). For the other direction,

$$\{H \leq G : H \text{ f.g.}\},$$

forms a directed system, as $H, K \leq \langle H \cup K \rangle \leq_f G$, so Theorem 8.9(iv) applies. $\square$

Corollary 8.11 (Soluble Amenable). Every countable soluble group is amenable, and thus sofic.

Proof. By Theorem 8.9(iii), it suffices to consider G countable abelian. By Corollary 8.10, we can assume G to be finitely generated, and thus by the Classification

$$G \cong \mathbb{Z}^d \times T,$$

for $|T| < \infty$. By Theorem 8.9(iii), we reduce to $\mathbb{Z}$ and T , which we already know are amenable by Examples 8.4 and 8.3. $\square$

Lecture 17

Aim. Prove Theorem 8.9 on closure properties of amenable groups.

Definition 8.12 (Left-Invariant ω -Additive Measure). A function

$$\mu : \mathcal{P}(G) \longrightarrow [0, 1],$$

is a **finitely additive probability measure** if

- (i) $\mu(G) = 1$,
- (ii) $\forall A, B \subseteq G, A \cap B = \emptyset \implies \mu(A \cup B) = \mu(A) + \mu(B)$.

We say μ is **left-invariant** if

$$\forall g \in G. \forall A \subseteq G. \mu(g \cdot A) = \mu(A).$$

Remark. A countable-additive left-invariant probability measure cannot exist, as any countable group can be expressed as a countable union of singletons.

Theorem 8.13 (Amenable Probability Measure). G is amenable if and only if G admits a left-invariant finitely additive probability measure.

Proof of 8.9. We have

- (i) $H \leq G$. By Theorem 8.13, G admits a l.i. f.a. probability measure μ_G . Let T be a right-transversal to H in G , and define

$$\mu_H(A) = \mu_G(A \cdot T).$$

- (ii) Through

$$\begin{aligned} \pi : G &\longrightarrow G/N \\ g &\longmapsto g \cdot N, \end{aligned}$$

we can define the pushforward measure

$$\begin{aligned} \mu_{G/N} &:= \pi_* \mu_G \\ \mu_{G/N}(A) &:= \mu_G(\pi^{-1}(A)). \end{aligned}$$

(iii) Given $\mu_N, \mu_{G/N}$. For $A \subseteq G$, define

$$\begin{aligned}\widehat{\delta}_A : G &\longrightarrow [0, 1] \\ g &\longmapsto \mu_N(N \cap g^{-1}A).\end{aligned}$$

For $n \in N$, we have

$$\begin{aligned}\widehat{\delta}_A(gn) &= \mu_N(N \cap (n^{-1}g^{-1}A)) \\ &= \mu_N(n^{-1}(N \cap g^{-1}A)) \\ &= \widehat{\delta}_A(g),\end{aligned}$$

so $\widehat{\delta}_A$ descends to

$$\delta_A : G/N \longrightarrow [0, 1].$$

We also note that (i) $\delta_G \cong 1$ and (ii) if $A \cap B = \emptyset$ then $\delta_{A \cup B} = \delta_A + \delta_B$. Define

$$\mu_G(A) := \int \delta_A d\mu_{G/N}.$$

By (i), $\mu_G(G) = 1$ and by (ii), we have for $A \cap B = \emptyset$

$$\mu_G(A \cup B) = \int (\delta_A + \delta_B) d\mu_{G/N} = \int \delta_A d\mu_{G/N} + \int \delta_B d\mu_{G/N}.$$

Finally, for $g, h \in G$,

$$\delta_{hA}(gN) = \mu_N(N \cap g^{-1}hA) = \delta(h^{-1}gN),$$

so

$$\begin{aligned}\mu_G(hA) &= \int \delta_{hA}(gN) d\mu_{G/N} = \int \delta_A(h^{-1}gN) d\mu_{G/N} \\ &= \int \delta_A(gN) d\mu_{G/N} \\ &= \mu_G(A),\end{aligned}$$

since $\mu_{G/N}$ is left-invariant.

(iv) $G = \bigcup_{i \in I} G_i$, $A \subseteq G$ finite, $\varepsilon > 0$. For $a \in A$, $\exists i_\alpha \in I$ with $a \in G_{i_\alpha}$, $\exists i \in I$ such that $\forall a \in A$, $G_i \geq G_{i_\alpha}$, so $A \subseteq G_i$. Since G_i is amenable, we have an (A, ε) -Følner set $F \subseteq G_i$.

□

The following proof will be foundationally sketchy. We will use AC in both directions.

Proof of 8.13. Assume G is amenable and let $A_n = A_n^{-1} \subseteq G$ finite, $A_n \subseteq A_{n+1}$, $\bigcup_n A_n = G$, $X := [0, 1]^{\mathcal{P}(G)}$. Then X is compact by Tychonoff's Theorem. Let

$$\begin{aligned}Y &= \{\nu \in X : \nu \text{ is a f.a. probability measure}\} \subseteq_C X \\ Y_n &= \{\nu \in Y : \forall a \in A_n, B \subseteq G, |\nu(B) - \nu(a \cdot B)| \leq \frac{1}{n}\} \subseteq_C X,\end{aligned}$$

| **Claim.** $\forall n \in \mathbb{N}, Y_n \neq \emptyset$.

Proof. Let $F_n \subseteq G$ be $(A_n, \frac{1}{2n})$ -Følner and define

$$\nu_n(B) = \frac{|B \cap F_n|}{|F_n|}.$$

We now wish to show that $\nu_n \in Y_n$. Certainly, ν_n is a f.a. prob. measure. For $a \in A_n$,

$$|F_n| \nu_n(aB) = |aB \cap F_n| = |B \cap a^{-1}F_n|.$$

Moreover,

$$\begin{aligned} |F_n| \cdot |\nu_n(B) - \nu_n(a \cdot B)| &\leq |B \cap (F_n \triangle a^{-1}F_n)| \leq |F_n \triangle a^{-1}F_n| \\ &\leq |(a^{-1}F_n) \setminus F_n| + |(aF_n) \setminus F_n| \\ &\leq \frac{1}{n} |F_n|. \end{aligned}$$

□

We note that $Y_{n+1} \subseteq Y_n$ for all $n \in \mathbb{N}$, and recall from point-set topology that, in a compact space, a descending sequence of non-empty closed sets has non-empty intersection. Hence we're done, since $\bigcap_n Y_n$ is the set of l.i. f.a. probability measure.

The other direction is non-examinable, the full argument is in the printed notes. Suppose G admits a l.i. f.a. probability measure, $A \subseteq G$ finite, $\varepsilon > 0$. We use a consequence of Hahn-Banach Separation Theorem: there exists

$$\varphi : G \longrightarrow [0, \infty),$$

with

$$(i) \quad \sum_{g \in G} \varphi(g) = 1,$$

$$(ii) \quad \forall a \in A, \sum_{g \in G} |\varphi(g) - \varphi(a^{-1}g)| < \frac{\varepsilon}{|A|}.$$

We insist on the additional assumption that $\varphi^{-1}((0, \infty))$ finite. We now use layer-cake representation: $\exists F_1 \subseteq F_2 \subseteq \dots \subseteq F_n \subseteq G$ finite, $C_i > 0$ such that

$$\varphi = \sum_{i=1}^n C_i \chi_{F_i}.$$

We now want to show by contradiction that there exist at least one i such that F_i is (A, ε) -Følner. We note that $\forall g \in G, \forall a \in A$,

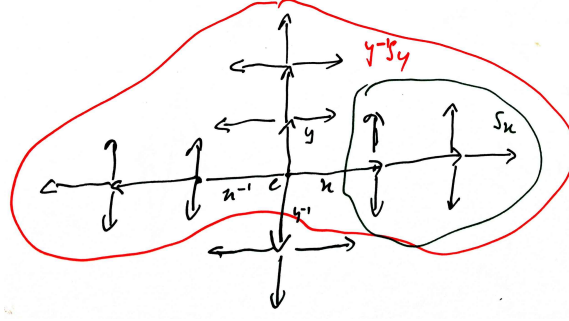
$$\chi_{aF_i \setminus F_i}(g) \leq |\chi_{F_i}(g) - \chi_{F_i}(a^{-1}g)|,$$

so by (ii)

$$\sum_i C_i |aF_i \setminus F_i| < \frac{\varepsilon}{|A|}.$$

By (i)

$$\sum_i C_i |F_i| = 1,$$

Figure 5: Decomposition of the Cayley graph of F_2 .

so

$$\sum_i C_i |aF_i \setminus F_i| < \frac{\varepsilon}{|A|} \sum_i C_i |F_i|.$$

By sum over $a \in A$,

$$\sum_i C_i \sum_a |aF_i \setminus F_i| < \varepsilon \sum_i C_i |F_i|.$$

By pigeon-hole, we have one i such that

$$\varepsilon |F_i| > \sum_a |aF_i \setminus F_i| \geq |A_i \setminus F_i|.$$

□

Lecture 18

Theorem 8.14. Non-abelian free groups are non-amenable.

Proof. Let X be a set such that $|X| \geq 2$. Let $x, y \in X, x \neq y$. Then the inclusion $\{x, y\} \hookrightarrow X$ induces an injective homomorphism $F_2 = F(\{x, y\}) \hookrightarrow F(X)$. Suffices to prove F_2 is not amenable.

Suppose F_2 was amenable, and thus did admit a l.i. f.a. probability measure. Let S_x be the set of reduced words in F_2 starting with x , i.e. the right cluster in Figure 5. Similarly, $S_{x^{-1}}$ for x^{-1} , S_y for y , $S_{y^{-1}}$ for y^{-1} . This decomposes

$$F_2 = S_x \amalg S_{x^{-1}} \amalg S_y \amalg S_{y^{-1}} \amalg \{e\}.$$

Letting $m_\bullet = \mu(S_\bullet)$ gives

$$1 = m_x + m_{x^{-1}} + m_y + m_{y^{-1}},$$

since $\mu(\{e\}) = 0$. We note that

$$\begin{aligned} x^{-1}S_x &= S_x \amalg S_y \amalg S_{y^{-1}} \amalg \{e\} \\ y^{-1}S_y &= S_x \amalg S_{x^{-1}} \amalg S_y \amalg \{e\}, \end{aligned}$$

thus by left invariance

$$\begin{aligned} m_x &= m_x + m_y + m_{y^{-1}} \\ m_y &= m_x + m_y + m_y \\ m_x + m_{x^{-1}} &= m_y + m_{y^{-1}} = 0, \end{aligned}$$

contradiction. □

Corollary 8.15. If G has a non-abelian free subgroup, then G is non-amenable.

Remark. This is the only criteria we have in this course to check non-amenability.

Conjecture (Von Neumann '29, Day '57). The converse holds.

Theorem 8.16. If $G \leq \text{GL}_d(K)$, with K a field and G finitely generated, then G satisfies conjecture above.

Proof. By consequence of the Tits' Alternative 7.24, if G has no non-abelian free subgroup, then G is virtually soluble, and thus amenable. $\square$

Theorem 8.17 (Olshanskii '80). The conjecture is false.

Proof. Beyond the course, constructs Tarski Monsters. See [KOO13]. $\square$

Definition 8.18 (Finitely G -Equidecomposable). Let G be a group acting on a set X , with $A, B \subseteq X$. We say A, B are **finitely G -equidecomposable** and write $A \sim_G B$, if there is some $n \in \mathbb{N}, g_1, \dots, g_n \in G$ such that

$$A = \coprod_{i=1}^n A_i, B = \coprod_{i=1}^n B_i,$$

such that $B_i = g_i A_i$ for all $1 \leq i \leq n$.

Remark. $\sim_G$ is an equivalence relation on $\mathcal{P}(X)$.

Remark. For $H \leq G$, if $A \sim_H B$, then $A \sim_G B$.

Definition 8.19 (G -Paradoxical). We say $Y \subseteq X$ is **G -paradoxical** if

$$\exists A, B \subseteq Y : A \cap B = \emptyset \wedge A \sim_G B \sim_G Y.$$

Example 8.20 (F_2). F_2 is F_2 -paradoxical, under left multiplication.

More generally, if $F_2 \leq G$, then G is G -paradoxical. For any T right transversal to F_2 in G , we have

$$\begin{aligned} A &= (S_x \coprod S_{x^{-1}}) \cdot T \\ B &= (S_y \coprod S_{y^{-1}}) \cdot T, \end{aligned}$$

and then proceed as in proof of Theorem 8.14.

Definition 8.21 (Paradoxical). We say that G is **paradoxical** if it is G -paradoxical, under left multiplication.

Theorem 8.22 (Tarski). G is non-amenable if and only if G is paradoxical.

Proof. The $\Leftarrow$ direction follows from an argument along the lines of Theorem 8.14, whereas the $\Rightarrow$ direction is beyond the scope of the course. $\square$

Definition 8.23 (Free Action). The action of G on X is **free** if

$$\forall x \in X, \text{Stab}_G(x) = \{e\}.$$

Fact. A group is free if and only if it acts freely on some tree.

Proof. See Serre's book on trees [Ser03]. $\square$

Theorem 8.24 (Hausdorff-Banach-Tarski Paradox). We have

- (i) $\mathbb{S}^2 = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| = 1\} \subseteq \mathbb{R}^3$ is $\text{SO}(3)$ -paradoxical.
- (ii) $B^3 = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| \leq 1\}$ is $\text{Isom}(\mathbb{R}^3)$ -paradoxical.

Remark. Weaker statement, the full Theorem allows for *decompositions without waste*.

Lemma 8.25. If G is paradoxical and G acts freely on X , then X is G -paradoxical.

Proof. Choose $Y \subseteq X$ containing exactly one point from each orbit of G . Then the sets

$$\{g \cdot Y : g \in G\},$$

partition X , by construction of Y .

Suppose $A \sim_G B \sim_G G$, with $A \cap B = \emptyset$, witnessed by $A = \coprod_{i=1}^n A_i, B = \coprod_{i=1}^n B_i$. Set

$$\overline{A}_i := \bigcup_{g \in A_i} g \cdot Y, \quad \overline{B}_i := \bigcup_{g \in B_i} g \cdot Y,$$

then

$$\overline{A} = \coprod_i \overline{A}_i \sim_G X \sim_G \coprod_i \overline{B}_i = \overline{B}.$$

$\square$

Theorem 8.26 (Hausdorff). $\exists H \leq \text{SO}(3)$ such that $H \cong F_2$.

Proof. Clever ping-pong. $\square$

Proposition 8.27. $\exists D \subseteq \mathbb{S}^2$ countable H -invariant such that $\mathbb{S}^2 \setminus D$ is H -paradoxical.

Proof. Recall $\forall g \in \text{SO}(3) \setminus \{I_3\}$, G has exactly 2 fixed points. Let

$$D := \{x \in \mathbb{S}^2 : \exists h \in H \setminus \{e\}. h \cdot x = x\}.$$

Then D is countable, since H is. Moreover, for $h, k \in H$, if $h \cdot x = x$, then

$$(khk^{-1})k \cdot x = h \cdot x,$$

so D is H -invariant. By construction, the action of H on $\mathbb{S}^2 \setminus D$ is free. $\square$

Lemma 8.28. Let G act on X , and $Y, Z \subseteq X$ with $Y \sim_G Z$. If Y is G -paradoxical, then so is Z .

Proof. Let

$$Y = \coprod_{i=1}^m Y_i, \quad Z = \coprod_{i=1}^m Z_i,$$

with $Z_i = g_i Y_i$. Since Y is G -paradoxical, we have some $A, B \subseteq Y$ with $A \cap B = \emptyset$ and $A \sim_G Y \sim_G B$. Set

$$A' := \coprod g_i(Y_i \cap A) \subseteq Z,$$

$$B' := \coprod g_i(Y_i \cap B) \subseteq Z.$$

Then $A' \cap B' = \emptyset$ and

$$A' \sim_G A \sim_G Y \sim_G B \sim_G B',$$

thus by transitivity,

$$A' \sim_G Z \sim_G B'.$$

$\square$

Proof of Banach-Tarski 8.24. Let $D \subseteq \mathbb{S}$ be as in Proposition 8.27. By Lemma 8.28, it suffices to show that $\mathbb{S}^2 \sim_{\text{SO}(3)} \mathbb{S}^2 \setminus D$.

Let $L \subset \mathbb{R}^3$ be a line through 0, such that $D \cap L = \emptyset$. Since D is countable, $\exists \Theta \in (0, 2\pi)$ such that if $\rho_\Theta \in \text{SO}(3)$ is a rotation about L by angle Θ ,

$$\forall n \in \mathbb{N}_{>0}. \rho_\Theta^n(D) \cap D = \emptyset.$$

Let

$$\overline{D} := \bigcup_{n=0}^{\infty} \rho_\Theta^n(D).$$

Then

$$\mathbb{S}^2 = \overline{D} \coprod (\mathbb{S}^2 \setminus \overline{D}) \sim_{\text{SO}(3)} \rho_\Theta(\overline{D}) \coprod (\mathbb{S}^2 \setminus \overline{D}) = \mathbb{S}^2 \setminus D.$$

$\square$

Theorem 8.29 (ES#3). B^3 is $\text{Isom}(\mathbb{R}^3)$ -paradoxical.

9 Residual Properties

Notation. We write $\mathcal{C}$ for a class of groups.

Definition 9.1 (Residually $\mathcal{C}$). We say G is **residually $\mathcal{C}$** if for all $g \in G \setminus \{e\}$, there exists some $Q \in \mathcal{C}$ and surjective homomorphism $\pi : G \twoheadrightarrow Q$ such that $\pi(g) \neq e_Q$.

Definition 9.2 (Fully Residually $\mathcal{C}$). We say G is **fully residually $\mathcal{C}$** if for all $A \subseteq G$ finite, there exists some $Q \in \mathcal{C}$ and surjective homomorphism $\pi : G \twoheadrightarrow Q$ such that $\pi|_A$ is injective.

Remark 9.3. If $G \in \mathcal{C}$, G is fully residually $\mathcal{C}$, and thus G is residually $\mathcal{C}$.

Remark 9.4. Suppose $\mathcal{C}$ is subgroup-closed, i.e. $Q \in \mathcal{C}, P \leq Q$ implies $P \in \mathcal{C}$. If G is (fully) residually $\mathcal{C}$ and $H \leq G$, then H is (fully) residually $\mathcal{C}$.

Lemma 9.5. Suppose $\mathcal{C}$ is closed under subgroups, finite direct products. Then G is residually $\mathcal{C}$ if and only if G is fully residually $\mathcal{C}$.

Proof. Assume G is fully residually $\mathcal{C}$. Given $e \neq g \in G$, it suffices to take $A = \{e, g\}$. Assume now G is residually $\mathcal{C}$ and consider

$$A := \{g_1, \dots, g_n\} \subseteq G.$$

For $1 \leq i < j \leq n$, we have $\pi_{ij} : G \twoheadrightarrow Q_{ij} \in \mathcal{C}$ such that $\pi_{ij}(g_i g_j^{-1}) \neq e$. Define the surjective homomorphism

$$\begin{aligned} \pi : G &\longrightarrow \prod_{1 \leq i < j \leq n} Q_{ij} \\ g &\longmapsto (\pi_{ij}(g))_{ij}. \end{aligned}$$

Then $\forall i \neq j. \pi(g_i) \neq \pi(g_j)$, and $\text{im}(\pi) \leq \prod_{ij} Q_{ij} \in \mathcal{C}$, so $\text{im}(\pi) \in \mathcal{C}$. □

Example 9.6. Classes of groups that satisfy the conditions of Lemma 9.5 are

- (i) Finite groups,
- (ii) Soluble groups,
- (iii) Amenable groups,
- (iv) Nilpotent groups,
- (v) Sofic groups

Proposition 9.7 (Sofic Residually Sofic). G is sofic iff it is G residually sofic.

Proof. One direction is given from Remark 9.3. Assume now that G is residually sofic. By Lemma 9.5, G is fully residually sofic. Let $A \subseteq G$ be finite, $\varepsilon > 0$. Then there exists Q sofic, $\pi : G \twoheadrightarrow Q$ such that $\pi|_A$ is injective. Let $\rho : \pi(A) \rightarrow \text{Symm}(n)$ be an $(\pi(A), \varepsilon)$ -representation. Then $\rho \circ \pi : A \rightarrow \text{Symm}(n)$ is an (A, ε) -representation. $\square$

Corollary 9.8. G residually finite $\implies G$ residually amenable $\implies G$ sofic.

9.1 Soficity of Free Groups

Aim. We want to show that free groups are sofic.

Idea. By Corollary 9.8, it suffices to show that free groups are residually finite. This is reasonable since free groups have lots of quotients.

Example 9.9. Finite groups are residually finite.

Example 9.10. Finitely generated abelian groups are residually finite.

Proof. Let G be finitely generated abelian. By the Classification, we have

$$G \cong \mathbb{Z}^d \times T,$$

with $|T| < \infty$. Take $e \neq g = (z_1, \dots, z_d, t) \in G$. If $t \neq e$, we take the surjective homomorphism

$$\begin{aligned} \pi_T : G &\longrightarrow T \\ g &\longmapsto t. \end{aligned}$$

Otherwise, we take the least i such that $z_i \neq 0$, and taking $n := |z_i| + 1$, we consider the surjective homomorphism

$$\begin{aligned} \pi_n : G &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ g &\longmapsto z_i \pmod{n}. \end{aligned}$$

$\square$

Example 9.11 (Non). Let $G = (\mathbb{Q}, +)$. For $\varphi : G \rightarrow Q$ with $|Q| < \infty$, $x \in \mathbb{Q}$,

$$\varphi(x) = \varphi\left(\frac{x}{|Q|}\right)^{|Q|} = e.$$

Example 9.12 (Non). Similarly, for the **Prufer group** for p prime

$$C_{p^\infty} = \left\{ \exp\left(\frac{2\pi im}{p^n}\right) : m, n \in \mathbb{N} \right\} \leq \mathbb{S}^1.$$

Example 9.13 (Non). G infinite simple.

Lecture 20

Theorem 9.14 (Malcev '40). Let K be a field, $G \leq \mathrm{GL}_d(K)$ finitely generated. Then G is residually finite.

Proof for $K = \mathbb{Q}$. Take $G = \langle S \rangle$, $S \subseteq \mathrm{GL}_d(\mathbb{Q})$ finite. There exists a finite set of primes $\{p_1, \dots, p_n\}$ such that all entries of all elements of $S \cup S^{-1}$ lie in $R = \mathbb{Z}[p_1^{-1}, \dots, p_n^{-1}]$.

Let $I_d \neq A \in \mathrm{GL}_d(R)$. Then there exists $1 \leq i < j \leq d$ such that $A_{ij} - \delta_{ij} \neq 0$. Write $A_{ij} - \delta_{ij} = \frac{r}{q} \in R$, with $r, q \in \mathbb{Z}$ coprime. Let p be a prime such that p and r are coprime, and $p \neq p_1, \dots, p_n$. Then

$$\begin{aligned} \pi : \mathbb{Z} &\longrightarrow \mathbb{Z}/p\mathbb{Z} \\ z &\longmapsto z \pmod{p}, \end{aligned}$$

extends to

$$\begin{aligned} \pi_p : R &\longrightarrow \mathbb{Z}/p\mathbb{Z} \\ z &\longmapsto z \pmod{p}, \end{aligned}$$

a ring homomorphism with $\pi_p(r/q) \neq 0$. This induces the group homomorphism

$$\begin{aligned} \pi_p : \mathrm{GL}_d(R) &\longrightarrow \mathrm{GL}_d(\mathbb{Z}/p\mathbb{Z}) \\ (x_{ij}) &\longmapsto (\pi_p(x_{ij})), \end{aligned}$$

and $\pi(A)_{ij} \neq \pi_p(I_d)_{ij}$, so $\pi_p(A) \neq I_d$. □

Remark. Malcev's Theorem 9.14 requires the following fact from ring theory to be proven in full generality.

Proposition 9.15 (Non-Examinable). Let R be a f.g. integral domain. Then

- (i) If R is a field, then R is finite.
- (ii) $\bigcap_{I \triangleleft_{\max} R} I = \{0\}$.

Corollary 9.16. Free groups are residually finite, and hence sofic.

Proof. Let X be any set and $e \neq w \in F(X)$. There exists $Y \subseteq X$ finite such that $w \in \langle Y \rangle \cong F(Y)$. Define the surjection

$$\begin{aligned} \varphi : F(X) &\longrightarrow F(Y) \\ \varphi(x) &= \begin{cases} x & \text{if } x \in Y \\ e & \text{if } x \notin Y \end{cases} \end{aligned}$$

Then $\varphi(w) \neq e$. Suffices to show that $F(Y)$ is residually finite. There exists an injection $Y \hookrightarrow \mathbb{Z}$ inducing injective homomorphism $F(Y) \hookrightarrow F(\mathbb{Z})$. By ES#1.4, $F(\mathbb{Z}) \hookrightarrow F(x, y)$. By ES#2.1, $F(x, y) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{GL}_2(\mathbb{Q})$. Done by Malcev's Theorem 9.14. □

Exercise (ES#4.9). Every f.p. residually finite group has solvable word problem.

9.2 Wreath Products

Aim. Construct f.g., perhaps even f.p., non-residually finite groups.

Notation. Let G, H be groups and denote

$$\bigoplus_G H := \{f \in H^G : |G \setminus f^{-1}(e_H)| < \infty\},$$

a group under pointwise multiplication. Define

$$\begin{aligned} \rho : G &\longrightarrow \text{Aut} \left(\bigoplus_G H \right) \\ \rho(g) : f &\longmapsto ({}_g f), \end{aligned}$$

with $({}_g f)(g') = f(g^{-1} \cdot g')$.

Definition 9.17 (Wreath Product). The **(regular restricted) wreath product** of H by G is

$$H \wr G := \left(\bigoplus_G H \right) \rtimes_\rho G.$$

More explicitly

$$(f_1, g_1) \cdot (f_2, g_2) = (f_1 \cdot ({}_{g_1} f_2), g_1 \cdot g_2).$$

Example 9.18 (T_2). Let T_2 be the binary rooted tree. By Proposition 7.32, we have

$$\text{Aut}(T_2) \cong \text{Aut}(T_2) \wr C_2.$$

Notation. For $g \in G, h \in H$, denote

$$(h_{(g)} : g' \mapsto h^{\delta_{g, g'}}) \in \bigoplus_G H.$$

For $T \subseteq H$, we write

$$T_{(g)} = \{t_{(g)} : t \in T\}.$$

Proposition 9.19. For $g, g' \in G$, we have

- (i) $H \cong H_{(g)} \leq \bigoplus_G H$,
- (ii) $gH_{(g')}g^{-1} = H_{(gg')}$,

⌊ (iii) $g \neq g' \implies [H_{(g)}, H_{(g')}] = \{e\}$.

Proof. Trivial. □

⌊ **Proposition 9.20.** If $G = \langle S \rangle, H = \langle T \rangle$, then $H \wr G = \langle S \cup T_{(e)} \rangle$. Thus if G, H are finitely generated, $H \wr G$ is finitely generated.

Proof. Let $f \in \bigoplus_G H, g \in G$. Let $X = G \setminus f^{-1}(e_H)$, which is finite. If $X = \emptyset$, then $f = e$ and thus $f \in \langle T_{(e)} \rangle$, so assume not. Then

$$f = \prod_{x \in X} f(x)_{(x)} = \prod_{x \in X} x f(x)_{(x)} x^{-1}.$$

Now $X \cup \{g\} \subseteq \langle S \rangle, f(x) \in \langle T \rangle$, and thus $f(x)_{(e)} \in \langle T_{(e)} \rangle$. Thus

$$f \cdot g = \left(\prod_{x \in X} x f(x)_{(e)} x^{-1} \right) \cdot g \in \langle S \cup T_{(e)} \rangle.$$

□

⌊ **Lemma 9.21.** $\pi : H \wr G \rightarrow Q$ homomorphism. If $\pi|_G$ is not injective, then

$$\bigoplus_G [H, H] \leq \ker(\pi).$$

Proof. We have

$$\bigoplus_G [H, H] = \langle [H, H]_{(g)} : g \in G \rangle.$$

Moreover, $[H, H]_{(g)} = g[H, H]_{(e)}g^{-1}$ and $[H, H]_{(e)} = \langle [h_1, h_2]_{(e)} : h_1, h_2 \in H \rangle$. Then it suffices to show that for all $h_1, h_2 \in H$,

$$[h_1, h_2]_{(e)} = [(h_1)_{(e)}, (h_2)_{(e)}] \in \ker(\pi).$$

Let $e \neq k \in \ker(\pi) \cap G$. Then

$$e = [(h_1)_{(k)}, (h_2)_{(e)}] = [k(h_1)_{(e)}k^{-1}, (h_2)_{(e)}],$$

and thus

$$\begin{aligned} e &= \pi(e) = \pi([k(h_1)_{(e)}k^{-1}, (h_2)_{(e)}]) \\ &= [\pi(k)\pi((h_1)_{(e)})\pi(k)^{-1}, \pi((h_2)_{(e)})] \\ &= \pi([(h_1)_{(e)}, (h_2)_{(e)}]). \end{aligned}$$

□

⌊ **Exercise (ES#4.6).** Let G, H be finitely generated, with G infinite and H non-trivial. Show that $H \wr G$ has exponential growth.

Theorem 9.22 (Grimberg '57). G, H groups, $H \wr G$ is residually finite iff

- (i) G, H are residually finite,
- (ii) G is finite or H is abelian.

Proof. Assume $H \wr G$ is residually finite. Then $G \leq H \wr G, H \cong H_{(e)} \leq H \wr G$, so G, H are residually finite. Towards a contradiction, suppose that $|G| = \infty$, and there are $h_1, h_2 \in H$ with $[h_1, h_2] \neq e$, Q finite group with $\pi : H \wr G \rightarrow Q$ such that $\pi([h_1, h_2]_{(0)}) \neq e$. But $\pi|_G$ is not injective, so by Lemma 9.21,

$$[h_1, h_2]_{(e)} \leq \bigoplus_G [H, H] \leq \ker(\pi),$$

contradiction. The other direction is on ES#4. □

Example 9.23. H finite non-abelian group. Then $H \wr \mathbb{Z}$ is f.g. non-residually f.p.

Definition 9.24 (Abels' Group). For p prime, a f.p. non-residually finite group is

$$\tilde{A}_p = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & p^n & * & * \\ 0 & 0 & p^m & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq \mathrm{GL}_4(\mathbb{Z}[p^{-1}]).$$

Lemma 9.25. Let

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mathbb{Z}[p^{-1}] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq \tilde{A}_p.$$

Then $H \leq Z(\tilde{A}_p)$, hence any subgroup of H is normal in $\tilde{A}_p$.

Definition 9.26 (p -Abels Group). The p -Abels' group is

$$\mathcal{A}_p := \tilde{A}_p / K,$$

where

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mathbb{Z} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq H.$$

Remark. $\mathcal{A}_p$ is soluble, since $\tilde{A}_p \leq B_4(\mathbb{Q})$, see ES#2.4.

Theorem 9.27. $\mathcal{A}_p$ is not residually finite.

Proof. Recall that by Example 9.12,

$$C_{p^\infty} \cong \mathbb{Z}[p^{-1}] / \mathbb{Z} \cong H / K \leq \mathcal{A}_p,$$

and C_{p^∞} is not residually finite. □

Theorem 9.28. $\mathcal{A}_p$ is finitely presented.

Proof. See (non-examinable) handout. It has 8 generators and circa 20 relations. $\square$

Theorem 9.29 (Baumslag+Solitar '62). The group

$$\langle x, y | xy^2x^{-1}y^{-3} \rangle,$$

is not residually finite.

Proof. See [BS62]. $\square$

9.3 Local Embeddings

Idea. The Theorem 8.5 uses too much: amenability requires homomorphisms from the whole group, whereas soficity is much more local.

Aim. We want to get some intermediate notion between amenability and soficity.

Definition 9.30 (Partial Homomorphism). Let G, Q groups, $A \subseteq G$. We say a function $\pi : A \rightarrow Q$ is a **partial homomorphism** if

$$\forall g, h \in A, gh \in A \implies \pi(gh) = \pi(g)\pi(h).$$

Remark. If $e_G \in A$, then $\pi(e_G) = \pi(e_G e_G) = \pi(e_G)^2$, and thus $\pi(e_G) = e_Q$.

Example 9.31 (Homomorphism Restriction). Let $\pi : G \rightarrow Q$ be a homomorphism. Then for any $A \subseteq G$, $\pi|_A : A \rightarrow Q$ is a partial homomorphism.

Assumption. Let $\mathcal{C}$ be a class of groups closed under subgroup and finite direct products.

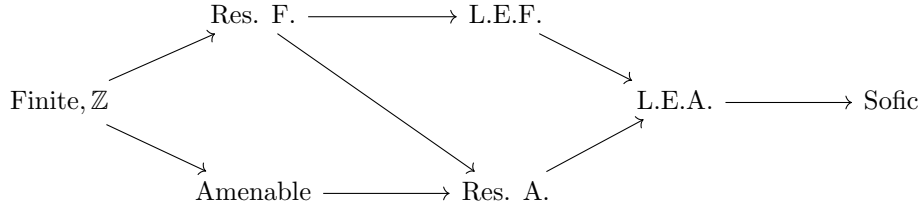
Definition 9.32 (L.E.). We say G is **locally embeddable into $\mathcal{C}$** (L.E. $\mathcal{C}$) if $\forall A \subseteq G$ finite, $\exists Q \in \mathcal{C}$ and an injective partial homomorphism $\pi : A \hookrightarrow Q$.

Exercise. Show that

- (i) If G is residually $\mathcal{C}$, then G is L.E. $\mathcal{C}$,
- (ii) If G is L.E.(L.E. $\mathcal{C}$), then G is L.E. $\mathcal{C}$,
- (iii) G is L.E. sofic iff G sofic (Proposition 9.7).
- (iv) G is L.E. $\mathcal{C}$ iff every finitely generated subgroup of G is L.E. $\mathcal{C}$ (Lemma 1.20).

Notation. We write L.E.F. for L.E. finite and L.E.A. for L.E. amenable.

Remark. Families of groups introduced so far:



Theorem 9.33. None of the arrows above can be reversed, or extra arrows added.

Proof. See ES#4.10. □

Theorem 9.34. If G, H are L.E.F. (L.E.A.), then $H \wr G$ is L.E.F. (L.E.A.).

Corollary 9.35. If G, H are L.E.F., with $|G| = \infty$, H non-abelian, then $H \wr G$ is L.E.F. but not residually finite.

Proof. Immediate from Theorem 9.34 and Theorem 9.22. □

Recall that $\mathcal{C}$ is closed under subgroups and finite direct products.

Theorem 9.36. If G is finitely presented L.E. $\mathcal{C}$, then G is residually $\mathcal{C}$.

Proof. Let $A \subseteq G$ be finite, then there exists a finite set X , $\rho : F(X) \twoheadrightarrow G$ a surjective homomorphism, and $R \subseteq \ker(\rho)$ finite with $\ker(\rho) = \langle\langle R \rangle\rangle^{F(X)}$. Let $n \in \mathbb{N}$ be such that $R \subseteq B_X(n)$ and $A \subseteq B_{\rho(X)}(n)$. Let $\pi : B_{\rho(X)}(n) \rightarrow Q$ be an injective partial homomorphism with $Q \in \mathcal{C}$. Define

$$\begin{aligned} \tilde{\pi} : F(X) &\longrightarrow Q \\ x &\longmapsto \pi(\rho(x)). \end{aligned}$$

Then the following diagram of partial homomorphisms commutes:

$$\begin{array}{ccc} & B_X(n) & \\ \rho|_{B_X(n)} \swarrow & & \searrow \tilde{\pi}|_{B_X(n)} \\ B_{\rho(X)}(n) & \xrightarrow{\pi} & Q \end{array}$$

so $\forall r \in R, \tilde{\pi}(r) = e$, hence $\tilde{\pi}$ descends to a homomorphism $\hat{\pi} : G \rightarrow Q$. For $w \in B_X(n)$,

$$\hat{\pi}(\rho(w)) = \tilde{\pi}(w) = \pi(\rho(w)).$$

□

Corollary 9.37. If G, H finitely generated L.E.F., $|G| = \infty$, H non-abelian. Then $H \wr G$ is finitely generated but not finitely presented.

Proof. Combine Proposition 9.20, Corollary 9.35, Theorem 9.36. $\square$

Lecture 22

Proof of 9.34. Proceed by steps.

Step 1: we reduce to $H \in \mathcal{C}$. Let $A_H \subseteq H$ finite be such that $\forall(f, g) \in H$, $\text{im}(f) \subseteq A_H$ and $e \in A_H$. Let $P \in \mathcal{C}$ and let $\pi : A_H \hookrightarrow P$ be an injective partial homomorphism. Define

$$\begin{aligned}\widehat{\pi} : A &\longrightarrow P \wr G \\ (f, g) &\longmapsto (\pi \circ f, g).\end{aligned}$$

Note that $e \in A_H$, $\pi(e_H) = e_P$, and thus the support

$$G \setminus (\pi \circ f)^{-1}(e_P) \subseteq G \setminus f^{-1}(e_H),$$

is finite, and thus $\pi \circ f \in \bigoplus_G P$.

Suppose $(f_1, g_1), (f_2, g_2) \in A$, with $(f_1, g_1) \cdot (f_2, g_2) \in A$,

$$\begin{aligned}\widehat{\pi}((f_1, g_1) \cdot (f_2, g_2)) &= \widehat{\pi}(f_1(g_1 f_2), g_1 g_2) \\ &= (\pi \circ (f_1 \cdot (g_1 f_2)), g_1 g_2),\end{aligned}$$

while

$$\widehat{\pi}(f_1, g_1) \widehat{\pi}(f_2, g_2) = ((\pi \circ f_1) \cdot (g_1 \pi \circ f_2), g_1 g_2).$$

Recall

$$(f_1 \cdot (g_1 f_2))(g) = f_1(g) \cdot f_2(g_1^{-1}g),$$

so

$$\begin{aligned}(\pi \circ (f_1 \cdot (g_1 f_2)))(g) &= \pi(f_1(g) \cdot f_2(g_1^{-1}g)) \\ &= \pi(f_1(g))\pi(f_2(g_1^{-1}g)) \\ &= (\pi \circ f_1)(g) \cdot (\pi \circ f_2)(g_1^{-1}g) \\ &= (\pi \circ f_1)(g) \cdot_{g_1} (\pi \circ f_2)(g),\end{aligned}$$

where $f_1(g) \cdot f_2(g_1^{-1}g), f_1(g), f_2(g_1^{-1}g) \in A_H$. By Exercise 9.3(ii), it suffices to show that $P \wr G$ is L.E.C.

Step 2: reduce now to $Q \in \mathcal{C}$. Let $A \subseteq P \wr G$ finite, $A_G \subseteq G$ finite be such that $\forall(f, g) \in A$:

- (i) $g \in A_{(g)}$,
- (ii) $G \setminus f^{-1}(e) \subseteq A_G$,
- (iii) $e \in A_G = A_G^{-1}$.

Let $Q \in \mathcal{C}$ and $\rho : A_G \hookrightarrow Q$ be an injective partial homomorphism, and take ∞ to be some object not in G . Define $\forall g \in G, g \cdot \infty = \infty$, to extend the action of G to $G \cup \{\infty\}$. Define $\rho' : Q \rightarrow G \cup \{\infty\}$ by

$$\rho'(q) = \begin{cases} g & \text{if } q = \rho(g) \\ \infty & \text{otherwise} \end{cases}$$

Set

$$\begin{aligned}\pi : A &\longrightarrow P \wr Q \\ (f, g) &\longmapsto (\widehat{f} \circ \rho', f(g)),\end{aligned}$$

where for $f \in P^G$, extend f to $\widehat{f} \in P^{G \cup \{\infty\}}$, via $\widehat{f}(\infty) = e$. We have that π is injective, since ρ is and $\widehat{f} = \widehat{f} \circ \rho' \circ \rho$. For $(f_1, g_1), (f_2, g_2) \in A$, we have

$$\begin{aligned}\pi(f_1, g_1)\pi(f_2, g_2) &= (\widehat{f}_1 \circ \rho', \rho(g_1))(\widehat{f}_2 \circ \rho', \rho(g_2)) \\ &= ((\widehat{f}_1 \circ \rho') \cdot (\rho(g_1)\widehat{f}_2 \circ \rho'), \rho(g_1)\rho(g_2)),\end{aligned}$$

while if $(f_1, g_1), (f_2, g_2) \in A$,

$$\begin{aligned}\pi((f_1, g_1) \cdot (f_2, g_2)) &= \pi(f_1 \cdot (g_1 f_2), g_1 g_2) \\ &= ((f_1 \cdot (g_1 f_2)) \circ \rho', \rho(g_1 g_2)) \\ &= ((\widehat{f}_1 \circ \rho') \circ ((g_1 f_2 \circ \rho')(q)), \rho(g_1)\rho(g_2)),\end{aligned}$$

so it suffices to show that for all $g \in Q$

$$((g_1 f_2) \circ \rho')(q) \tag{*}$$

$$= (\rho(g_1)(\widehat{f}_2 \circ \rho'))(q). \tag{*}$$

Note that by property (ii), $G \setminus f_1^{-1}(e) \subseteq A_G$ and $(f_1 \cdot (g_1 f_2), g_1 g_2) \in A$, so

$$G \setminus (g_1 f_2)^{-1}(e) = G \setminus (g_1(f_2^{-1}(e))) \subseteq A_G,$$

thus

$$G \setminus f_2^{-1}(e) \subseteq A_G \cap g_1^{-1}A_{(g)}. \tag{†}$$

Hence, we have (a) if $(*)$ is non-trivial

$$((g_1 f_2) \circ \rho')(q) \neq e,$$

then $q = \rho(x)$ with $f_2(g_1^{-1}x) \neq e$, so $x \in A_G \cap g_1 A_G$, while (b) if $(*)$ is non-trivial

$$(\rho(g_1)(\widehat{f}_2 \circ \rho'))(q) \neq e,$$

then

$$\rho(g_1)^{-1}q = \rho(z),$$

with $f_2(z) \neq e$, so $z \in A_G \cap g_1^{-1}A_G$ by (†). Then write $z = g_1^{-1}y, y \in A_G$, so

$$\rho(z) = \rho(g_1^{-1}y) = \rho(g_1)^{-1}\rho(y),$$

and $q = \rho(y)$. Hence $y = g_1 z \in A_{(g)} \cap g_1 A_G$. We conclude that if $q \notin \rho(A_G \cap g_1 A_G)$, then $(*) = (\star) = e$.

If $q = \rho(x)$ with $x = g_1 w \in A_G \cap g_1 A_G$, so that $q = \rho(g_1 w) = \rho(g_1)\rho(w)$. Then $(*) =_{g_1} f_2(x) = f_2(w)$, hence

$$(\star) = (\widehat{f}_2 \circ \rho) = (\rho(g_1)^{-1}q) = (\widehat{f}_2 \circ \rho')(\rho(w)) = f_2(w).$$

Step 3: If P, Q are finite, then $P \wr Q$ finite, and if P, Q are amenable, $P \wr Q$ is amenable, by Theorem 8.9. $\square$

10 Space of Marked Groups

Definition 10.1 (Space of d -Marked Groups). Fix $d \in \mathbb{N}$ and set $X_d = \{x_1, \dots, x_d\}$. We call

$$\mathcal{G}_d := \{N \triangleleft F(X_d)\},$$

the **space of d -marked groups**.

Remark. The function

$$\begin{aligned} \nu : \mathcal{G}_d \times \mathcal{G}_d &\longrightarrow \mathbb{N} \cup \{\infty\} \\ \nu(N, M) &\longmapsto \max\{n \in \mathbb{N} : N \cap B_{X_d}(n) = M \cap B_{X_d}(n)\}, \end{aligned}$$

induces a metric on $\mathcal{G}_d$ by

$$d(N, M) = 2^{-\nu(N, M)}.$$

Theorem 10.2 (Grigorchuk '84). $\mathcal{G}_d$ is compact.

Sketch of Proof. Extend d , i.e. ν , to $\mathcal{P}(F(X_d)) = 2^{F(X_d)}$, which is compact by Tychonoff's. Moreover, the topology induced by d is compatible with the topology on $2^{F(X)}$. Then it suffices to check "being a normal subgroup" is a closed condition. $\square$

Idea (Alternative Perspective). Let G be a d -generated group with generators $s_1, \dots, s_d \in G$. Thus we have a surjective homomorphism

$$\begin{aligned} \pi_{\underline{s}} : F(X_d) &\longrightarrow G \\ x_i &\longmapsto s_i, \end{aligned}$$

where $\underline{s} = (s_1, \dots, s_d)$. This gives an element $N_{\underline{s}} = \ker(\pi_{\underline{s}}) \in \mathcal{G}_d$,

Definition 10.3 (d -Marking). We call $\underline{s}$ a **d -marking** on G , writing $[G, \underline{s}] := \ker(\pi_{\underline{s}})$.

Remark. We have the following correspondence: Conversely, given $N \triangleleft F(X_d)$, letting $G = F(X_d)/N$, $s_i = x_i N$, we have $N = [G, \underline{s}]$.

$$\begin{aligned} \{G \text{ } d\text{-generated}\} &\longleftrightarrow \{N \triangleleft F(X_d)\} \\ G, (s_1, \dots, s_d) &\longmapsto [G, \underline{s}] \triangleleft F(X_d) \\ F(X_d)/N, (x_i N) &\longleftarrow N \triangleleft F(X_d) \end{aligned}$$

Remark. We have

- (i) $[G, \underline{s}] = [H, \underline{t}] \iff \ker(\pi_{\underline{s}}) = \ker(\pi_{\underline{t}})$, i.e. iff there is a commutative diagram

$$\begin{array}{ccc} & F(X_d) & \\ \pi_{\underline{s}} \swarrow & & \searrow \pi_{\underline{t}} \\ G & \xrightarrow{\quad} & H \\ \xleftarrow{\quad} & & \xrightarrow{\quad} \end{array}$$

iff there is an isomorphism

$$\begin{aligned} G &\longrightarrow H \\ s_i &\longmapsto t_i. \end{aligned}$$

- (ii) There is an isometric embedding

$$\begin{aligned} \iota : \mathcal{G}_d &\longrightarrow \mathcal{G}_{d+1} \\ [G, (s_1, \dots, s_d)] &\longmapsto [G, (s_1, \dots, s_d, e)]. \end{aligned}$$

Proposition 10.4. F.p. d -generated groups form a dense subspace

$$\{[G, \underline{s}] : G \text{ is f.p.}\} \subseteq \mathcal{G}_d.$$

Proof. Let $N \in \mathcal{G}_d$. For $n \in \mathbb{N}$, let $R_n = N \cap B_{X_d}(n)$. Then $N_n := \langle \langle R_n \rangle \rangle^{F(X_d)} \leq N$, and $N \cap B_{X_d}(n) = R_n = N_n \cap B_{X_d}(n)$, so $d(N, N_n) \leq 2^{-n}$.

Moreover, $G_n = F(X_d)/N_n$ is finitely presented and

$$N_n = [G_n, (x_i N_n)_i].$$

□

Theorem 10.5. Let $\mathcal{C}$ be a class of groups closed under subgroups and finite direct products, and define

$$\mathcal{C}_d = \{[H, \underline{t}] \in \mathcal{G}_d, H \in \mathcal{C}\} \subseteq \mathcal{G}_d.$$

For G a d -generated group, T.F.A.E.

- (i) G is L.E. $\mathcal{C}$,
- (ii) $\forall \underline{s}$ d -markings on G , $[G, \underline{s}] \in \overline{\mathcal{C}_d}$,
- (iii) $\exists \underline{s}$ d -marking on G , such that $[G, \underline{s}] \in \overline{\mathcal{C}_d}$,

Proof. (ii) $\implies$ (iii) trivial.

(iii) $\implies$ (i): let $A \subseteq G_d$ be finite, $S = \{s_1, \dots, s_d\}$. Then $A \subseteq B_S(n)$ for some $n \in \mathbb{N}$. Let $[Q, \underline{t}] \in \mathcal{C}_d$ with

$$\ker(\pi_{\underline{s}}) \cap B_{X_d}(2n) = \ker(\pi_{\underline{t}}) \cap B_{X_d}(2n) = R_n.$$

Then $\pi_{\underline{s}}, \pi_{\underline{t}}$ descend to

$$\begin{array}{ccc}
 & & G \\
 & \nearrow \pi_{\underline{s}} & \\
 F(X_d) & \xrightarrow{\pi} & F(X_d)/\langle\langle R_n \rangle\rangle^{F(X_d)} \\
 & \searrow \pi_{\underline{t}} & \\
 & & H
 \end{array}$$

and $\tilde{\pi} = \bar{\pi}_{\underline{s}}|_{B_{\pi}(X_d)(n)}$ is an injective partial homomorphism. We can check that

$$\tilde{\pi}^{-1} : B_S(n) \longrightarrow F(X_d)/\langle\langle R_n \rangle\rangle^{F(X_d)} = \langle X_d | R_n \rangle,$$

is an injective partial homomorphism.

Then $\bar{\pi}_{\underline{t}} \circ \tilde{\pi}^{-1} : B_S(n) \rightarrow Q$ is an injective partial homomorphism.

(i) $\implies$ (ii): $\forall n \in \mathbb{N}, \exists Q \in \mathcal{C}, \rho : B_S(n) \rightarrow Q$ injective partial homomorphism

$$\begin{array}{ccc}
 & G \supseteq B_S(n) & \\
 \nearrow \pi_{\underline{s}} & \downarrow \rho & \\
 F(X_d) & & Q \\
 \searrow \pi_{\underline{t}} & & \downarrow
 \end{array}$$

WLOG, we take $Q = \langle \rho(S) \rangle$, as $\mathcal{C}$ is closed under subgroups. Then $\underline{t} = (\rho(s_1), \dots, \rho(s_d))$ is a d -marking on Q , with $\pi_{\underline{s}}(B_{X_d}(n)) = B_S(n)$, $\rho(e_G) = e_Q$, hence $B_{X_d}(n) \cap \ker(\pi_{\underline{s}}) \subseteq \ker(\pi_{\underline{t}})$. Conversely, suppose now that $w \in B_{X_d}(n) \cap \ker(\pi_{\underline{t}})$. Then

$$\rho(\pi_{\underline{s}}(w)) = \pi_{\underline{t}}(w) = e_Q,$$

as $\pi_{\underline{s}}(w) \in B_S(n)$, but ρ is one-to-one, so $\pi_{\underline{s}}(w) = e_G$, so

$$B_{X_d}(n) \cap \ker(\pi_{\underline{t}}) \subseteq \ker(\pi_{\underline{s}}).$$

□

Corollary 10.6. If there is a non-sofic group, then there is a f.p. non-sofic group.

Proof. Suppose that every f.p. group is sofic. Let $\mathcal{C}$ be the class of subgroups of f.p. groups, which is closed under subgroups and finite direct products.

Then every group in $\mathcal{C}$ is sofic. By Proposition 10.4 and Theorem 10.5, every f.g. group is L.E.C, and thus L.E. sofic, and thus sofic by Exercise 9.3. Finally, by Lemma 1.20, every group is sofic. □

Corollary 10.7. If G is d -generated, non-amenable and $[G, \underline{s}] \in \mathcal{G}_d$ is an isolated point, then G is not L.E.A.

Remark. By Proposition 10.4, marked f.p. d -generated groups are dense in $\mathcal{G}_d$. Hence, if G is not f.p., then $[G, \underline{s}]$ is not isolated, as it would be the limit point of a sequence of f.p. groups.

Example 10.8. G f.p. residually finite d -generated group, $|G| = \infty$, e.g. $G = \mathbb{Z}$. By Theorem 10.5, for any d -marking $\underline{s} \in G^d$, $[G, \underline{s}]$ is an accumulation point of marked finite groups in $\mathcal{G}_d$. So f.p. is not enough for being isolated.

Definition 10.9 (Discriminating Set). $D \subseteq G \setminus \{e\}$ is a **discriminating set** for G if, for any $\{e\} \neq N \triangleleft G$, $D \cap N \neq \emptyset$.

Theorem 10.10 ([dCGP07]). Let G be a d -generated group. TFAE:

- (i) $\forall \underline{s}$ d -markings of G , $[G, \underline{s}]$ is isolated in $\mathcal{G}_d$,
- (ii) $\exists \underline{s}$ a d -marking of G , $[G, \underline{s}]$ is isolated in $\mathcal{G}_d$,
- (iii) G is finitely presented and has a finite discriminating set.

Definition 10.11 (Isolated Group). We call G **isolated** if it satisfies the conditions of Theorem 10.10.

Proof of 10.10. (i) $\implies$ (ii) direction: immediate.

(ii) $\implies$ (iii) direction: suppose (iii) fails and let $\underline{s} \in G^d$ be a d -marking. If G is not f.p., then by Proposition 10.4, $[G, \underline{s}]$ is not isolated. Suppose G has no finite discriminating set. Then, letting $S = \{s_1, \dots, s_d\}$, $B_S(n) \setminus \{e\}$ is not a discriminating set, so $\exists \{e\} \neq N \triangleleft G$ with $B_S(n) \cap N = \{e\}$. Letting

$$\begin{aligned} \pi : G &\longrightarrow G/N \\ g &\longmapsto gN, \end{aligned}$$

we have

$$\ker(\pi) \cap B_S(n) = \{e\},$$

and so $\pi(\underline{s}) = (\pi(s_i))_i$ is a d -marking on G/N and

$$0 < d([G, \underline{s}], [G/N, \pi(\underline{s})]) \leq 2^{-n},$$

contradicting (ii).

(iii) $\implies$ (i) direction: G f.p. with $\underline{s}$ a d -marking on G , $S = \{s_1, \dots, s_d\}$. Recall by Proposition 2.29, $\exists R_{\underline{s}} \subseteq \ker(\pi_{\underline{s}})$ finite such that $\ker(\pi_{\underline{s}}) = \langle \langle R_{\underline{s}} \rangle \rangle^{F(X_d)}$.

Let $n_0 \in \mathbb{N}$ with $R_{\underline{s}} \subseteq B_{X_d}(n_0)$. For $n \geq n_0$, if $d([G, \underline{s}], [H, \underline{t}]) \leq 2^{-2n}$, then $R_{\underline{s}} \subseteq \ker(\pi_{\underline{t}})$, so $\ker(\pi_{\underline{s}}) \leq \ker(\pi_{\underline{t}})$.

Then $\pi_{\underline{t}}$ factors as

$$F(X_d) \xrightarrow{\pi_{\underline{s}}} G \xrightarrow{\theta} H$$

$$x_i \longmapsto s_i \longmapsto t_i$$

Moreover, $\ker(\pi_{\underline{s}}) \cap B_{X_d}(2n) = \ker(\pi_{\underline{t}}) \cap B_{X_d}(2n)$, so $\theta|_{B_S(n)}$ is injective.

Let $D \subseteq G \setminus \{e\}$, be a finite discriminating set for G . For n sufficiently large, $D \subseteq B_S(n)$, so $\ker(\theta) \cap D = \emptyset$, hence $\ker(\theta) = \{e\}$ and θ is an isomorphism and $[G, \underline{s}] = [H, \underline{t}]$. $\square$

Example 10.12. Every finite group is isolated.

Example 10.13. Every finitely presented simple group is isolated.

We haven't seen them in the course, but they exist.

Example 10.14 (ES#4). For any prime p , the p -Abels group is isolated.

Remark. Start of non-syllabus material:

Theorem 10.15 (Cornulier [Cor11]). There exists a non-L.E.A. sofic group.

Sketch of Proof. Take

$$\tilde{\mathcal{C}}_p = \left\{ \begin{pmatrix} a & b & * & * & * \\ c & d & * & * & * \\ 0 & 0 & p^n & * & * \\ 0 & 0 & 0 & p^m & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} \leq \mathrm{GL}_5(\mathbb{Z}[p^{-1}]),$$

and quotient

$$\mathcal{C}_p := \tilde{\mathcal{C}}_p / K, K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \tilde{\mathcal{C}}_p.$$

Then

(i) $\mathcal{C}_p$ is fg. isolated, by mimicking the proof for Abels' group.

(ii) Let

$$A := \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $B := A^\top$. Then $\langle A, B \rangle \cong F_2$, by ES#2.1.

(iii) Proof of Soficity.

□

Theorem 10.16 (Elek+Szabo [ES03]). If $N \triangleleft G$, with N sofic, G/N amenable, then G is sofic.

Question (Open). Does $N, G/N$ being sofic imply G sofic? What if N is finite?

Theorem 10.17 (Elek+Szabo [ES03]). G, H sofic, then $G * H$ sofic.

Question (Open). Is the amalgamated free product of two sofic groups sofic? What about the amalgamated product of two free groups?

Question (Open). Does there exist a

- (i) Finitely presented infinite torsion group?
- (ii) A finitely presented group of intermediate growth?
- (iii) A finitely presented group of intermediate growth slower than $e^{\sqrt{n}}$?
- (iv) A group of intermediate growth slower than Grigorchuk's group?

Definition 10.18 (Law). We say $e \neq w \in F(X_d)$ is a **law** for G if, for every homomorphism $\pi : F(X_d) \rightarrow G$, $w \in \ker(\pi)$.

Some groups don't have laws, i.e. $F(X_2)$. The study of groups with laws is somewhat of a group theoretic analogue of algebraic geometry.

Question (Open). Let G be residually finite and with a law. Must G be amenable?

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