

UNIVERSITY OF
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MATHEMATICS TRIPOS

Part III

Large Cardinals

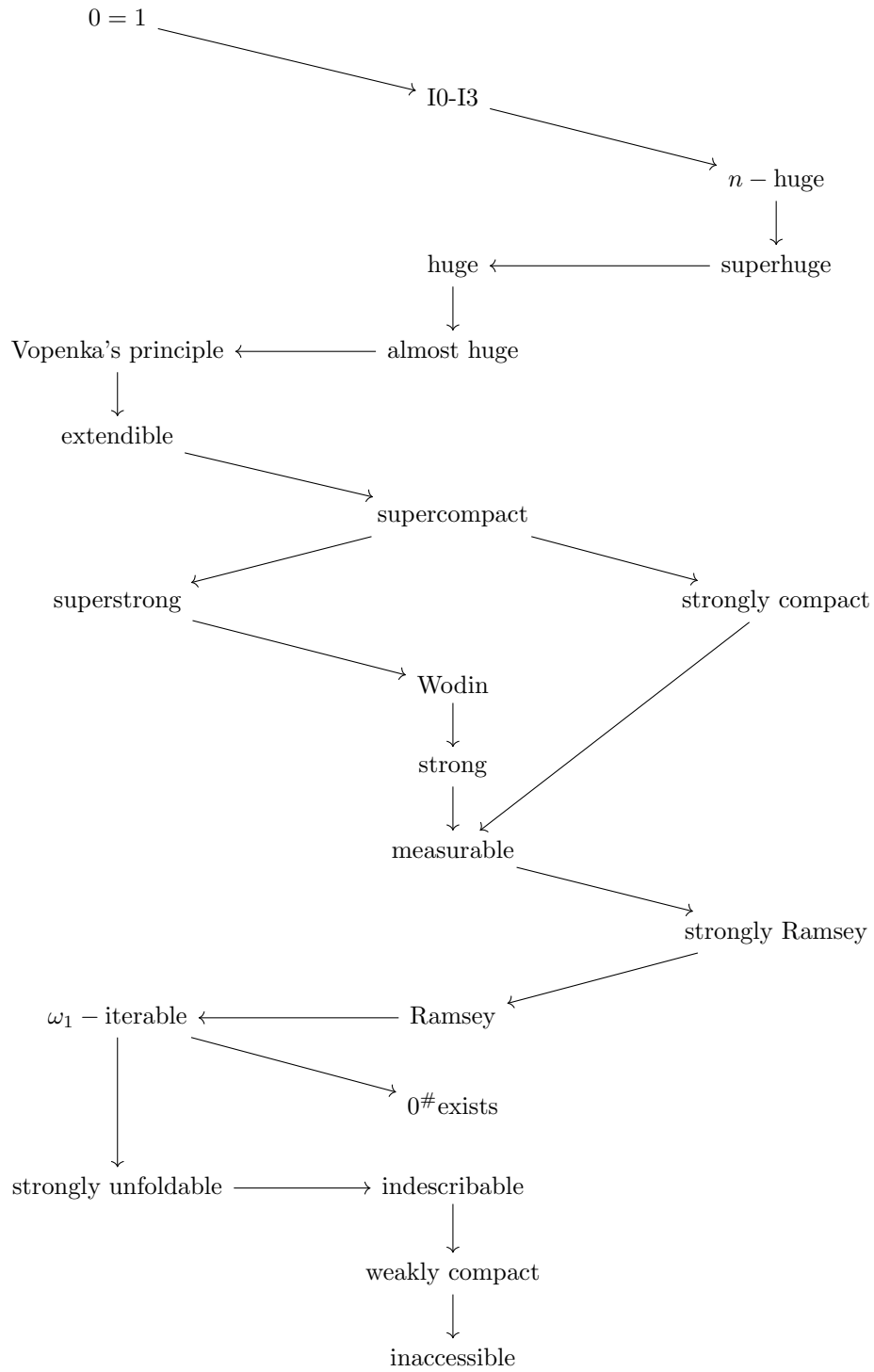
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0 Introduction

The following definitions and facts should be familiar from an introductory course on set theory: a cardinal κ is called **regular** if every unbounded subset of κ has cardinality κ ; successor cardinals, i.e., cardinals of the form $\aleph_{\alpha+1}$, are always regular; the usual limit cardinals, e.g., $\aleph_\omega$, $\aleph_{\omega+\omega}$, or $\aleph_{\omega_1}$, are not.

Thus, the following is a natural question:

“Are there any uncountable regular limit cardinals?”

If they exist, they must be very large, in particular, much larger than any of the mentioned limit cardinals.

It turns out that this question is intricately connected with the incompleteness phenomenon in set theory: if there is an uncountable regular limit cardinal, then there is a model of ZFC; therefore, ZFC is consistent, and hence (by Gödel’s Second Incompleteness Theorem) ZFC cannot prove the existence of these cardinals (unless, of course, it is inconsistent).

Regular limit cardinals (a.k.a. **weakly inaccessible cardinals**) are the smallest examples of set-theoretic notions called **large cardinals**: cardinals with properties that imply that they must be very big and whose existence cannot be proved in ZFC.

In this course, we shall get to know a number of these large cardinals, study their behaviour, observe consequences of their existence for set theory, and develop techniques to determine the logical strength of large cardinals (the so-called **consistency strength hierarchy**). In modern set theory, large cardinals are used as the standard way to calibrate logical strength of extensions of ZFC.

References are [Jec03, Kan03b]. The first one is very discursive, gives history of the results, whereas the second one is more encyclopedic.

Lecture 1

The chart reported on first page comes from [Kan03b]. It represents the largely-linear consistency strength hierarchy. The goal of the course is to show some results about the arrows.

1 Defining Large Cardinals

We start off with a non-definition:

Definition 1.1 (Non). A **large cardinal property** is a property of a cardinal κ that implies that

- (i) κ is very big,
- (ii) κ is so big that ZFC cannot prove existence of such cardinals.

A **large cardinal** is a cardinal with a LCP and a **large cardinal axiom** is an axiom of the form

$$\exists \alpha : \Phi(\alpha),$$

where Φ is an LCP.

Remark. It is somewhat unclear what the link between (i) and (ii) is.

We will work towards defining weakly and strongly inaccessibles, at the very end of the diagram. The latter are also called inaccessibles.

1.1 First Non-Example

Let us analyse a property that implies κ is very big.

Definition 1.2 (Normal ordinal operations). A **normal ordinal operation** is an operation

$$F : \text{Ord} \longrightarrow \text{Ord},$$

that is monotone, i.e.

$$\alpha < \beta \implies F(\alpha) < F(\beta),$$

and continuous w.r.t. order topology, in which successor points are isolated and for every limit ordinal λ we have

$$F(\lambda) = \bigcup_{\alpha < \lambda} F(\alpha).$$

Remark. Note that F is not a function, as it is a class operating like a function.

We have the following result from ES#1 of Part II Logic and Set-Theory:

Theorem 1.3. Every normal ordinal operation has arbitrarily large f.p. $\xi = F(\xi)$.

Question. What is the smallest fixed point of $\alpha \mapsto \omega^\alpha$? What is the next smallest?

Answer. The following is a normal ordinal operation:

$$\begin{aligned} \aleph_0 &:= \omega \\ \aleph_{\alpha+1} &:= \aleph_\alpha^+ \\ \aleph_\lambda &:= \bigcup_{\alpha < \lambda} \aleph_\alpha, \end{aligned}$$

By Theorem 1.3, there must be arbitrarily large fixed points $\kappa = \aleph_\kappa$, called **Aleph fixed points**.

These must be very large: they must be limit cardinals since κ is a cardinal, κ is a limit ordinal so $\aleph_\kappa = \kappa$ must be a limit cardinal. By looking at first candidates:

$$\begin{aligned} \aleph_\omega &\neq \omega \text{ as } \omega \neq 0 \\ \aleph_{\omega_1} &\neq \aleph_1 \text{ as } \omega_1 \neq 0 \\ \aleph_{\aleph_\omega} &\neq \aleph_\omega \text{ as } \aleph_\omega \neq \omega, \end{aligned}$$

we see that the gap between indices is growing quite fast.

The limit is known to be the limit of $\aleph_{\aleph_n} = \bigcup_{n \in \mathbb{N}} \aleph_{\aleph_n}$, is much bigger than all the $\aleph$ -s. However, this cardinal is not too big for ZFC, so it cannot be a large cardinal in the sense of property (ii). This is because if $\text{AFC}(\kappa) \iff \kappa$ is an Aleph fixed point, then $\text{AFP} \iff \exists k : \text{AFC}(k)$, thus $\text{ZFC} \vdash \text{AFC}$, by Theorem 1.3.

1.2 Second Non-Example

We need the following notion:

Definition 1.4 (Cofinal). Let λ be a limit ordinal. Then $C \subseteq \lambda$ is called **cofinal** or **unbounded** in λ if

$$\begin{aligned} & \forall \alpha < \lambda \exists \gamma \in C : (\alpha < \gamma) \\ \iff & \lambda = \bigcup C. \end{aligned}$$

Remark. Clearly, λ is unbounded in λ .

Definition 1.5 (Cofinality). The **cofinality** of λ is

$$\text{cf}(\lambda) := \min\{|C| : C \text{ cofinal in } \lambda\}.$$

Remark. By previous remark, $\text{cf}(\lambda) \leq |\lambda|$, thus $\text{cf}(\lambda)$ is a cardinal.

Remark. For a cardinal κ , we have $\text{cf}(\kappa) \leq \kappa$.

Definition 1.6 (Regular/Singular). We say κ is

- (i) **Regular** if $\text{cf}(\kappa) = \kappa$,
- (ii) **Singular** if $\text{cf}(\kappa) < \kappa$.

Exercise (ES#2 L&ST). Show that ω_1 is not a countable union of countable ordinals, i.e. $\text{cf}(\omega_1) = \omega_1$ or $\aleph_1$ is regular.

Example ($\aleph_\omega$ Singular). We have that $\aleph_\omega$ is singular, since

$$\aleph_\omega = \bigcup \{\aleph_n : n \in \mathbb{N}\},$$

is cofinal, thus $\text{cf}(\aleph_\omega) \leq \aleph_0$. Moreover, we have $\text{cf}(\aleph_\omega) = \aleph_0$, since finite sets cannot be cofinal.

Remark. In general, if $\kappa = \aleph_\lambda$ is a limit cardinal, i.e. if λ is a limit ordinal, then

$$\kappa = \aleph_\lambda = \bigcup \{\aleph_\alpha : \alpha < \lambda\},$$

thus $\text{cf}(\aleph_\lambda) \leq \text{cf}(\lambda)$.

We discussed that $\aleph_1$ is regular. We can generalise this to

Theorem 1.7 (ZFC). Every successor cardinal is regular.

Proof. The proof makes heavy use of AC. Let $\kappa = \aleph_{\alpha+1}$, by defn, if $\xi < \kappa$, there exists a surjection

$$\pi_\xi : \aleph_\alpha \twoheadrightarrow \xi,$$

for each ξ . To move towards a contradiction, assume

$$\kappa = \bigcup C,$$

with C of size $\leq \aleph_\alpha$, so fix a surjection $\psi : \aleph_\alpha \twoheadrightarrow C$.

Define

$$\begin{aligned} \pi : \aleph_\alpha \times \aleph_\alpha &\longrightarrow \aleph_{\alpha+1} \\ (\gamma, \delta) &\longmapsto \pi_{\psi(\gamma)}(\delta), \end{aligned}$$

which is a surjection. But by Hessenberg's theorem, this can't be as $\aleph_\alpha \times \aleph_\alpha$ bijects with $\aleph_\alpha$ so we found a surjection $\aleph_\alpha \twoheadrightarrow \aleph_{\alpha+1}$. Contradiction. $\square$

Remark. For $\aleph_1$ the argument is roughly the same, except the map $\omega \times \omega \rightarrow \aleph_1$ does not require Hessenberg's theorem.

So we have the following situation:

- Every successor cardinal is regular.
- No successor cardinal is singular.
- All the concrete examples of singular limit cardinals are the ones we came up with.

Question. Are there any regular limit cardinals?

2 Inaccessibles

Definition 2.1 (Weak inaccessibles). A cardinal is **weakly inaccessible** if it is a regular limit cardinal.

To show that this implies (i), we can piggy back on discussion of $\aleph$ fixed points.

Proposition 2.2. Weak inaccessible cardinals are Aleph fixed points.

Proof. Let κ be a weak inaccessible, then it is a limit cardinal, thus $\kappa = \aleph_\lambda$ for some λ limit ordinal. But by previous argument $\text{cf}(\kappa) = \text{cf}(\aleph_\lambda) \leq \text{cf}(\lambda) \leq \lambda$. Thus if $\lambda < \kappa$, then $\text{cf}(\kappa) \leq \lambda < \kappa$, so κ is singular.

This proves $\lambda = \kappa$, making κ an Aleph fixed point. $\square$

We cannot immediately prove (ii), i.e.

Proposition 2.3. $\text{ZFC} \not\vdash \exists$ weak inaccessibles.

But we can show the slightly strengthened result:

Definition 2.4 (Strongly inaccessible). We say a cardinal κ is **(strongly) inaccessible** if for each $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Remark. We compare with previous definition: instead of requiring being closed under the cardinal successor operation, we are requiring strongly inaccessible cardinals to be closed under the cardinal powerset operation, which is stronger.

Lecture 2

Example ($\kappa = \aleph_\kappa$). Aleph fixed points satisfy (i), as clearly very big, but no (ii).

Example (Inaccessibles). Weakly inaccessible are regular limit cardinals and strongly inaccessible are regular strong limit cardinals.

Aim. Want to prove that ZFC $\not\models \exists$ inaccessible.

Remark. Unless obviously ZFC is inconsistent.

Notation. $\text{ICC}(\kappa) \iff \kappa$ is a regular and strong limit, and $\text{IC} \iff \exists \kappa. \text{ICC}(\kappa)$.

Definition 2.5 (Cumulative Hierarchy). **Von Neumann/cumulative hierarchy**

$$\begin{aligned} V_0 &:= \emptyset \\ V_{\alpha+1} &:= \mathcal{P}(V_\alpha) \\ V_\lambda &:= \bigcup_{\alpha < \lambda} V_\alpha, \end{aligned}$$

λ limit ordinal.

Proposition 2.6. We have

- All V_α are transitive sets,
- $\alpha \leq \beta \implies V_\alpha \subseteq V_\beta$ (cumulative),
- $V_\alpha \cap \text{Ord} = \alpha$.

Theorem 2.7 (L&ST). The axiom of Foundation/Regularity is equivalent to

$$\forall x \exists \alpha : x \in V_\alpha.$$

Definition 2.8 (Mirimanoff rank).

$$\rho(x) := \min\{\alpha : x \in V_{\alpha+1}\}.$$

Idea. Sort of birthday of sets.

Proposition 2.9. In ZFC, every set has a Mirimanoff rank (by Foundation) and we can prove statements about all sets by induction over the rank.

Reminder ES#4 L&ST, questions 10 and 11 about V_ω and $V_{\omega+\omega}$.

Remark. Most ZFC axioms are of the form $\forall x \exists y$, so proving such an axiom in $(V_\alpha, \in)$ is the same as proving a bound on ρ for y in terms of x .

Exercise (ES#4.10). If λ is a limit ordinal then $(V_\lambda, \in)$ satisfies all axioms of ZFC, with the exception of Infinity and Replacement (we could prove Replacement but relies on finiteness).

Exercise (ES#4.11). If $\lambda > \omega$, then Infinity is true in $(V_\lambda, \in)$.

Summary:

Proposition 2.10. If $\lambda > \omega$ is a limit ordinal, then $(V_\lambda, \in) \models \text{ZC}$, i.e. Zermelo set theory with choice, all of ZFC without Replacement.

Remark. Replacement really fails. Otherwise if held for $\lambda = \omega + \omega$, ZFC would be inconsistent by Gödel.

Theorem 2.11 (Hausdorff). If κ is inaccessible, then $V_\kappa \models \text{ZFC}$.

Remark. By previous result, suffices to show $V_k \models \text{Replacement}$.

Corollary 2.12. If ZFC is consistent then $\text{ZFC} \not\models \text{IC}$.

Proof. Suppose $\text{ZFC} \models \text{IC}$. By Hausdorff 2.11, $\text{IC} \implies \exists M (M \models \text{ZFC})$, but by Gödel's Completeness Theorem, $\exists M (M \models \text{ZFC}) \iff \text{Cons}(\text{ZFC})$.

Thus this implies that $\text{ZFC} \models \text{IC} \implies \text{ZFC} \models \text{Cons}(\text{ZFC})$, thus by Gödel's Second Incompleteness Theorem this implies ZFC is inconsistent. $\square$

Remark. There is a way to avoid using Gödel's second.

Lemma 2.13. If $\text{IC}(\kappa)$ and $\alpha < \kappa$, then $|V_\alpha| < \kappa$.

Proof. By induction:

$$|V_0| = 0 < \kappa.$$

Suppose now that $|V_\alpha| < \kappa$, then

$$\begin{aligned} |V_{\alpha+1}| &= |P(V_\alpha)| \\ &= 2^{|V_\alpha|} < \kappa, \end{aligned}$$

where the last follows by κ strong limit. Let λ be a limit. Suppose that for all $\alpha < \lambda$, $|V_\alpha| < \kappa$,

$$|V_\lambda| = \left| \bigcup_{\alpha < \lambda} V_\alpha \right| \leq \bigcup_{\alpha < \lambda} |V_\alpha|,$$

thus $\{|V_\alpha| : \alpha < \lambda\}$ is a set of ordinals in a set with size $\leq |\lambda| < \kappa$.

Thus by Regularity: $\bigcup_{\alpha < \lambda} |V_\alpha|$ is bounded below κ . So $|V_\lambda| < \kappa$. $\square$

Remark. We mostly used Infinity and Regularity, so important characterisation of inaccessibles.

Corollary 2.14. If $\text{IC}(\kappa)$ and $x \in V_\kappa$, then $|x| < \kappa$.

Proof. If $x \in V_\kappa$, there is a $\alpha < \kappa$ s.t. $x \in V_\alpha$. By transitivity, $x \subseteq V_\alpha$, $|x| \leq |V_\alpha| < \kappa$. $\square$

Proof of 2.11. We already have ZC, so it suffices to prove Replacement, which is everyone's least favourite axiom. We can actually strengthen Replacement to the following principle: for all $F : V_k \rightarrow V_k$ and all $x \in V_k$ the image

$$F[x] := \{F(y) : y \in x\} \in V_k.$$

This is what we would want the replacement axiom to say.

The reason why we didn't say it like this in Replacement is that it is not expressible in first order logic. Replacement is the restriction of $F[x]$ formula to F that are definable over V_k by a first order formulae. This is why we call it Second Order Replacement (SOR).

Clearly SOR implies Replacement, if V_κ satisfies SOR, we're good (note that we can't say $V_\kappa \models \text{SOR}$ since SOR is not first order).

Onto the proof of SOR: fix $F : V_\kappa \rightarrow V_\kappa$, fix $x \in V_\kappa$. Need to show $F[x] \in V_\kappa$.

$$F[x] = \{F(y) \in V_\kappa : y \in x\},$$

so $\rho(F(y)) < \kappa$,

$$C := \{\rho(F(y)) : y \in x\} \subseteq \kappa.$$

But $|C| \leq |x| < \kappa$ by Corollary, so by Regularity of κ , C is bounded, say by α .

So $F[x] \subseteq V_\alpha$ and $F[x] \in V_{\alpha+1} \subseteq V_\kappa$. $\square$

Remark. We can prove $\text{ZFC} \not\models \text{IC}$ from Hausdorff without using Gödel's Second Incompleteness Theorem.

First need to show that if κ is inaccessible and $\lambda < \kappa$ then

$$\text{IC}(\lambda) \iff V_\kappa \models \text{IC}(\lambda).$$

See ES#1.

Now assume $\text{ZFC} \vdash \text{IC}$. Thus pick κ_0 be the least inaccessible.

By Hausdorff 2.11, $V_{\kappa_0} \models \text{ZFC}$, and since $\text{ZFC} \vdash \text{IC}$, $V_{\kappa_0} \models \text{IC}$.

Therefore, there is $\lambda < \kappa_0$ such that

$$V_{\kappa_0} \models \text{IC}(\lambda) \iff \text{IC}(\lambda),$$

by previous, in contradiction with minimality of κ_0 .

Remark. This proof is slick and elegant. The proof of Hausdorff's Theorem seems to suggest that everything fits so nicely that it should be an equivalence:

Definition 2.15 (Wordly). We say that κ is **worldly** if $V_\kappa \models \text{ZFC}$.

Remark. The name worldly was not introduced until 15 years ago.

Remark. Hausdorff says every inaccessible is worldly.

Question. Are these notions equivalent? Is Hausdorff's Theorem an equivalence?

Answer. No and Yes.

Notions of worldly and inaccessibility are not equivalent, but there is an equivalence in Hausdorff: the equivalence is not with worldliness but rather with the stronger property SOR (see ES#1 and next lectures).

No: if κ is inaccessible, there are many worldly cardinals which are not inaccessible!

Lecture 3

2.1 Accessible Worldlies

Aim. We wish to show that *inaccessible* is strictly stronger than *worldly*.

Theorem 2.16. If κ is inaccessible and $\alpha < \kappa$, then there is a λ s.t. $\alpha < \lambda < \kappa$ and λ is worldly.

Corollary 2.17. If κ_0 is the least inaccessible, then there are accessible worldly cardinals.

Remark. The role of α is to show that there are actually many of them, as there are actually κ -many of them.

First we have to show that $V_\kappa \models \text{ZFC} \implies \kappa$ is a cardinal.

Proposition 2.18 (Worldlies are Cardinals). Every worldly κ is a cardinal.

Proof. First we observe that κ must be a limit ordinal, since every

$$V_{\alpha+1} \models \text{there is a largest ordinal},$$

so $V_{\alpha+1} \not\models \text{ZFC}$.

Suppose that κ is not a cardinal. Then there must be some $\lambda \in V_\kappa$ that bijects $\pi : \lambda \leftrightarrow \kappa$ with κ . We construct a well-order on λ of order type κ in V_κ by

$$R := \{(\alpha, \beta); \pi(\alpha) < \pi(\beta)\}.$$

By construction, π is an isomorphism between (λ, R) and $(\kappa, \in)$. But $R \in V_{\lambda+1}$, so $(\lambda, R) \in V_{\lambda+3} \subseteq V_\kappa$.

Since $V_\kappa \models \text{ZFC}$, V_κ satisfies the representation theorem for well-orders: every well-order is isomorphic to a unique ordinal. Thus $\kappa \in V_\kappa$. Contradiction. $\square$

Remark. A minor improvement of that argument shows that: every worldly α is a limit cardinal. See ES#1.

We will now prove existence of accessible worldlies.

Definition 2.19 (Elementary Equivalence). Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$ -structures. We write $\mathcal{M} \equiv \mathcal{N}$ and read $\mathcal{M}$ is **elementarily equivalent** to $\mathcal{N}$ if for all $\mathcal{L}$ -sentences σ , $\mathcal{M} \models \sigma \iff \mathcal{N} \models \sigma$.

Definition 2.20 (Elementary Substructure). For $\mathcal{M} \subseteq \mathcal{N}$, we say $\mathcal{M}$ is an **elementary substructure** of $\mathcal{N}$, write $\mathcal{M} \preceq \mathcal{N}$, iff for all $\mathcal{L}$ -formulas φ with n free variables and all $\vec{a} \in M^n$, $\mathcal{M} \models \varphi(\vec{a}) \iff \mathcal{N} \models \varphi(\vec{a})$.

Remark. $\mathcal{M} \equiv \mathcal{N}$ is not equivalent to $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N} \preceq \mathcal{M}$. In fact, $\mathcal{M} \preceq \mathcal{N} \implies \mathcal{M} \equiv \mathcal{N}$ and in general is much stronger.

This is because $\mathcal{M} \equiv \mathcal{N}$ only talks about only $\mathcal{L}$ -sentences.

Remark. Our construction of non-standard models while keeping the theory the same is an application of this.

Main tool for elementary substructures

Proposition 2.21 (Tarski-Vaught Test). Suppose $\mathcal{M}$ is a substructure $\mathcal{N}$. Then $\mathcal{M}$ is elementary if and only if, for any formula $\varphi(v, \vec{w})$ and $\vec{a} \in \mathcal{M}$, if there is $b \in N$ such that $\mathcal{N} \models \varphi(b, \vec{a})$ then there is $c \in M$ such that $\mathcal{N} \models \varphi(c, \vec{a})$.

Proof. Induction on complexity of formulas. Atomic formulas and propositional connectives are preserved for all substructures, so trivial. The only missing step is $\varphi \mapsto \exists x \varphi$ and that step needs precisely the assumption given in TVT. See [Mar02] for full proof. $\square$

Idea. This means that it suffices to check that existence $\mathcal{L}$ -formulas are preserved.

Remark. Since ZFC is a theory (i.e. it consists of sentences), if $V_\kappa \models \text{ZFC}$ and $V_\lambda \equiv V_\kappa$, then $V_\lambda \models \text{ZFC}$. Similarly, if $V_\lambda \preceq V_\kappa$, then $V_\lambda \models \text{ZFC}$.

In the language of large cardinals, if κ is worldly and $V_\lambda \preceq V_\kappa$, then λ is worldly.

Theorem 2.22. If κ is inaccessible and $\alpha < \kappa$, then there is a λ such that $\alpha < \lambda < \kappa$ and λ is worldly.

Rephrased and made stronger as:

Theorem 2.23. If κ is inaccessible, then $\{\lambda < \kappa; V_\lambda \preceq V_\kappa\}$ is unbounded in κ .

Corollary 2.24. Since κ is regular, there are κ many worldly cardinals below κ .

Corollary 2.25. The least worldly cannot be the least inaccessible.

Proof of 2.23. To use the Tarski-Vaught Test 2.21, need to find witnesses for formulae $\exists x\varphi(x, \vec{a})$ where $\vec{a} \in V_\lambda$, if $V_\kappa \models \exists x\varphi(x, \vec{a})$. We will collect witnesses in an ω -iteration to make the union an elementary substructure by Tarski-Vaught Test 2.21. Fix $\alpha < \kappa$, try to find λ s.t. $\alpha < \lambda < \kappa$ and $V_\lambda \preceq V_\kappa$.

We start the ω -iteration at $\alpha+1$ by letting $\alpha_0 := \alpha+1$. Suppose α_i is already defined and $\alpha_i < \kappa$. By Lemma 2.13, we know that $|V_{\alpha_i}| < \kappa$, since strongly inaccessible satisfy ZFC. Therefore we have for the set of finite sequences from V_{α_i}

$$|V_{\alpha_i}^{<\omega}| < \kappa.$$

Furthermore, the set Fml of formulas in the language of set theory is countable, hence

$$|\text{Fml} \times V_{\alpha_i}^{<\omega}| < \kappa.$$

Suppose $\vec{a} \in V_{\alpha_i}^m$ and $\varphi \in \text{Fml}$ with $m+1$ free variables

$$V_\kappa \models \exists x\varphi(x, \vec{a}).$$

This can be either true or false, but TVT makes no requirements about the false case. Thus suppose it is true, and consider for example the sentence without parameters "there exists a large cardinal". We define w by

$$w(\varphi, \vec{a}) := \begin{cases} 0 & \text{if } V_\kappa \not\models \exists x\varphi(x, \vec{a}) \\ \gamma & \text{if } V_\kappa \models \exists x\varphi(x, \vec{a}) \implies \exists \gamma \exists c \in V_\gamma : V_\kappa \models \varphi(c, \vec{a}) \end{cases}$$

Since $|\text{Fml} \times (V_{\alpha_i})^{<\omega}| < \kappa$, then the set

$$W := \{w(\varphi, \vec{a}); \varphi \in \text{Fml}, \vec{a} \in (V_{\alpha_i})^m\},$$

where φ has at least $m+1$ free variables. Since $|W| < \kappa$, by regularity of κ it must be bounded, say by β , and thus we can define

$$\alpha_{i+1} := \max(\alpha+1, \beta)$$

We now define

$$\lambda := \bigcup_{i \in \mathbb{N}} \alpha_i,$$

which clearly has $\text{cf}(\lambda) = \aleph_0$ and thus $\lambda \neq \kappa$.

We now want to show that $V_\lambda \preceq V_\kappa$ by TVT. Suppose

$$V_\kappa \models \exists x\varphi(x, \vec{a}),$$

where $\vec{a} \in (V_\lambda)^m$ and φ has $m+1$ free variables. Since λ is a limit ordinal, and $\lambda = \bigcup_{i \in \mathbb{N}} \alpha_i$ we find sufficiently large N such that $\vec{a} \in (V_{\alpha_N})^m$. Thus

$$V_\kappa \models \exists x\varphi(x, \vec{a}),$$

means that we're in case (ii) of the construction in the step $\alpha_N \mapsto \alpha_{N+1}$. Therefore, there is a witness $c \in V_{\alpha_{N+1}}$ such that

$$V_\kappa \models \varphi(c, \vec{a}).$$

This shows TVT. □

Remark. As we have seen in the proof, $\text{cf}(\lambda) = \aleph_0$. This means that

(i) The least such worldly cardinal has cofinality $\aleph_0$.

(ii) The proof shows that

$$W^{\aleph_0} := \{\lambda < \kappa; V_\lambda \preceq V_\kappa, \text{cf}(\lambda) = \aleph_0\},$$

is unbounded in κ .

Remark. If a worldly cardinal is regular, then it is weakly inaccessible, by ES#1.

This means that there is no hope to show that any of these $\lambda < \kappa$ are regular because if the GCH holds, for all $\lambda < \kappa$, $2^\lambda = \lambda^+$, thus there is no difference between weak and strong inaccessibles. Thus in ZFC+GCH there are inaccessible cardinals with no regular worldlies below.

Question. Can we get worldlies with higher cofinality, e.g. $\text{cf}(\lambda) = \aleph_1, \aleph_2, \dots$?

Remark. Recall the two conditions for being a large cardinal: (i) very big and (ii) existence is not provable. By Second Gödel Incompleteness, (ii) is immediate for worldlies. Arguable whether there're big enough.

Lecture 4

2.2 Basic Model Theory

Proposition 2.26. If $\mathcal{M}_0 \preceq \mathcal{N}, \mathcal{M}_1 \preceq \mathcal{N}, \mathcal{M}_0 \subseteq \mathcal{M}_1$, then $\mathcal{M}_0 \preceq \mathcal{M}_1$.

Proof. Follows from definition. □

Idea. There are lots of elementary substructures of V_κ . Beneath the least inaccessible, have lots of worldlies. Thus Theorem 2.23 implies that if κ is inaccessible, we have lots of pairs (λ, λ') such that $\lambda < \lambda' < \kappa$ with $V_\lambda \preceq V_{\lambda'}$.

Definition 2.27 (Elementary Chain). Let $(L, <)$ be a total order and $(M_l; l \in L)$ a family of structures indexed by L . We say $(M_l; l \in L)$ an **elementary chain** if for all $l < l', M_l \preceq M_{l'}$.

Lemma 2.28 (Tarski's Chain). If $(M_l; l \in L)$ is an elementary chain and $M := \bigcup_{l \in L} M_l$, then for each $l \in L, M_l \preceq M$.

Proof. On ES#1, similar to proof of the Tarski-Vaught Test 2.21. □

Theorem 2.29. If κ is inaccessible and $\mu < \kappa$ is regular, then there is $\lambda < \kappa$ such that λ is worldly and $\text{cf}(\lambda) = \mu$.

Remark. In general, cannot guarantee that for some μ , we have $\mu = \lambda$.

Proof. Let $W := \{\lambda \in \kappa; V_\lambda \preceq V_\kappa\}$. Then $|W| = \kappa$, so take its increasing enumeration given by λ_α being the α -th element of W . By Proposition 2.26, if $\alpha < \beta$, then $V_{\lambda_\alpha} \preceq V_{\lambda_\beta}$. So, for an arbitrary $X \subseteq \kappa$ we have that $\{V_{\lambda_\alpha}; \alpha \in X\}$ forms an elementary chain. So, consider $\{V_{\lambda_\alpha}; \alpha < \mu\}$. Then

$$\bigcup_{\alpha < \mu} V_{\lambda_\alpha} = V_\lambda,$$

where $\lambda = \bigcup_{\alpha < \mu} \lambda_\alpha$ as limit.

By Tarski's Chain Lemma 2.28, we get $V_{\lambda_\alpha} \preceq V_\lambda$ for any $\alpha < \mu$. In particular, $V_\lambda \models \text{ZFC}$, so λ is worldly. Since μ was regular and $\lambda = \bigcup_{\alpha < \mu} \lambda_\alpha$, we have $\text{cf}(\lambda) = \mu$. $\square$

3 Weakly Inaccessible Cardinals

Remark. We're still yet to prove that their existence cannot be proved in ZFC. We're not going to do a full proof of this, as it rests of a fundamental theorem of existence of inner models of GCH, i.e. the proof of $\text{Cons}(\text{ZFC}) \rightarrow \text{Cons}(\text{ZFC} + \text{GCH})$ by Gödel, which will be blackboxed.

3.1 Models of Set Theory

Proposition 3.1. Suppose $\mathcal{M}, \mathcal{N}$ are models of set theory such that $M \subseteq N$ and they use the same element relation, i.e. $(M, \in), (N, \in)$. Then $(M, \in)$ is a substructure of $(N, \in)$.

Proof. Atomic formulas will remain true if we go from $\mathcal{M}$ to $\mathcal{N}$, as well as if going from $\mathcal{N}$ to $\mathcal{M}$ and parameters lie in M . Since they're only of the form $x \in y$ or $x = y$. $\square$

Definition 3.2 (Transitive Model). We say that $\mathcal{M}$ is **transitive** in $\mathcal{N}$ if $x, y \in N, x \in M, y \in x$ implies $y \in M$.

Remark. This means that M is a transitive subclass of N .

Example. If $\lambda < \kappa$, then V_λ is transitive in V_κ .

Remark. Model of set theory is vague, not even specified ZFC, could be very weak theory such as finite set theory.

Definition 3.3 (Bounded Quantification). Let Λ be a class of formulas. We say

(i) Λ is **closed under propositional connectives** if

$$\varphi, \psi \in \Lambda \implies \varphi \wedge \psi, \psi \wedge \varphi, \neg \varphi \in \Lambda,$$

(ii) Λ is **closed under bounded quantification** if

$$\varphi \in \Lambda \implies \exists x(x \in y \wedge \varphi) \in \Lambda,$$

where y serves as a bound for the variable x .

Definition 3.4 (Δ_0). The class Δ_0 is the smallest class of formulas that contains the atomic formulas and is both closed under propositional connectives and bounded quantification.

Definition 3.5 (Absoluteness). We say a formula φ is **absolute between** $\mathcal{M}$ and $\mathcal{N}$ if for $\vec{a} \in M^n$

$$\mathcal{M} \models \varphi(\vec{a}) \iff \mathcal{N} \models \varphi(\vec{a}).$$

Theorem 3.6. Suppose $\mathcal{M}$ is transitive in $\mathcal{N}$ and φ is Δ_0 formula with n free variables. Then φ is absolute between $\mathcal{M}$ and $\mathcal{N}$.

Proof. Δ_0 is defined by recursion (closure), so we can prove this by induction by proving that the construction steps of Δ_0 preserve absoluteness. The construction steps are atomic, propositional and bounded quantification.

- (i) Atomic is preserved by all substructures.
- (ii) Propositional connectives follows by definition of the semantics of $\wedge, \vee, \neg$.
- (iii) For bounded quantification, our induction hypothesis is that φ is absolute between $\mathcal{M}$ and $\mathcal{N}$, and we want to show that $\exists x(x \in y \wedge \varphi)$ is absolute. Suppose $\mathcal{M} \models \exists x(x \in a \wedge \varphi)$. Then by definition, we can find $b \in M$ such that

$$\begin{aligned} & \mathcal{M} \models b \in a \wedge \varphi \\ \implies & \mathcal{M} \models b \in a \text{ and } \mathcal{M} \models \varphi \\ \implies & \mathcal{N} \models b \in a \text{ and } \mathcal{N} \models \varphi \\ \implies & \mathcal{N} \models b \in a \wedge \varphi \\ \implies & \mathcal{N} \models \exists x(x \in y \wedge \varphi). \end{aligned}$$

Suppose now that $\mathcal{N} \models \exists x(x \in a \wedge \varphi)$ and thus by definition, we find $b \in N$ such that

$$\begin{aligned} & \mathcal{N} \models b \in a \wedge \varphi \\ & \mathcal{N} \models b \in a \text{ and } \mathcal{N} \models \varphi \\ & \mathcal{M} \models b \in a \text{ and } \mathcal{M} \models \varphi \\ & \mathcal{M} \models \exists x(x \in a \wedge \varphi), \end{aligned}$$

where $a \in M, b \in a \implies b \in M$ by transitivity.

□

Question. Why does this result help us?

Answer. Lots of important notions in set theory are defined by Δ_0 formulas, or are equivalent to Δ_0 formulas in ZFC.

Remark. We haven't yet used any properties of set theory. They will become important in applications.

Example (Being an Ordinal). Recall the official definition: x is an ordinal if and only if x is a transitive set such that $(x, \in)$ is a well-order.

This is not a Δ_0 formula, as quantifier is not bounded. However, the foundation axiom says that $\in$ is well-founded, so we don't need to require well-order, which gets us

$$\text{ZFC} \vdash x \text{ is an ordinal} \iff x \text{ is transitive and } (x, \in) \text{ is a total order.}$$

Transitivity expands out to

$$\forall u, v (v \in x \wedge u \in v \rightarrow v \in x),$$

and total order to

$$\begin{aligned} \forall u, v, w (u, v, w \in x \rightarrow (u \in v \wedge v \in w \rightarrow v \in w) \\ \wedge u \not\in u \wedge (u \in v \vee v \in u \vee u = v)). \end{aligned}$$

As a consequence, we get that if $\mathcal{M}$ is transitive in $\mathcal{N}$ and $\mathcal{M} \models \text{ZFC}, \mathcal{N} \models \text{ZFC}$, then

$$\forall x \in M, \mathcal{M} \models x \text{ is an ordinal} \iff \mathcal{N} \models x \text{ is an ordinal.}$$

Warning. This does not mean $\text{Ord} \cap M = \text{Ord} \cap N$, e.g. V_λ, V_κ with $\lambda < \kappa$. This is an additional property which we'll discuss in Lecture 5.

Lecture 5

Question. Why does $\text{ZFC} \not\models \text{IC}$?

Assumption. Have $M \subseteq N$, $(M, \in) \models \text{ZFC}, (N, \in) \models \text{ZFC}$, $\mathcal{M}$ is transitive in $\mathcal{N}$.

Example. $V_\lambda \subseteq V_\kappa \implies V_\lambda$ is transitive in V_κ , e.g. κ inaccessible and λ is the constructed worldly cod. Below κ , then $(V_\lambda, \in) \models \text{ZFC}$ and $(V_\kappa, \in) \models \text{ZFC}$.

Question. Can we given an example where $\mathcal{M}$ is not transitive in $\mathcal{N}$?

Answer. Construction has nothing to do with large cardinals but it is useful to have.

Remark. Recall that Δ_0 formulas are absolute between transitive models.

Example (Non). Trivial example of non-preservation of Δ_0 formulas for non-transitive models. Let $(N, \in) \models \text{ZFC}$ be a model of ZFC and $M := \{0, 2\}$.

Clearly $(M, \in)$ is not a model of ZFC. Have the following Δ_0 formula:

$$\begin{aligned} \varphi(x) &= \text{there is exactly one element of } x \\ \varphi(x) &= \exists u (u \in x) \wedge \forall v (v \in x \rightarrow \forall w (w \in x \rightarrow v = w)). \end{aligned}$$

This is Δ_0 , $(M, \in) \models \varphi(2)$ but $(N, \in) \not\models \varphi(2)$.

Later we'll see that a non-transitive $M \subseteq N$ such that $(M, \in) \models \text{ZFC}$.

Definition 3.7 (Σ_1, Π_1). A formula is called Σ_1 if it is of the form $\exists x\psi$, and Π_1 if it is of the form $\forall x\psi$, where ψ is a Δ_0 formula.

Remark. They are not absolute between transitive models, but are halfway there.

Definition 3.8 (Downwards/Upwards Absolute). A formula is called **downwards absolute** between $\mathcal{M}$ and $\mathcal{N}$ if $\forall \vec{x} \in M^n$, we have

$$\mathcal{N} \models \varphi(\vec{x}) \implies \mathcal{M} \models \varphi(\vec{x}),$$

and **upwards absolute** if

$$\mathcal{N} \models \varphi(\vec{x}) \iff \mathcal{M} \models \varphi(\vec{x}).$$

Proposition 3.9. Σ_1 formulas are upwards absolute and Π_1 formulas are downwards absolute for transitive models of ZFC.

Proof. This follows directly from the fact that Δ_0 formulas are absolute and $M \subseteq N$:

$$\begin{aligned} (M, \in) \models \exists x\psi &\implies \exists a \in M, (M, \in) \models \psi(a), \\ &\implies \exists a \in N, (M, \in) \models \psi(a), \\ &\implies (N, \in) \models \exists x\psi. \end{aligned}$$

□

Remark. Don't actually need 'of ZFC', but need strong theory so that enough formulas are Δ_0 . Δ_0 is entirely syntactic. For instance, the formula that defines an ordinal is not Δ_0 , but it is ZFC-equivalent to Δ_0 .

Example 3.10. The following properties are described in ZFC by a Π_1 -formula

- (i) κ is a cardinal,
- (ii) κ is a regular cardinal,
- (iii) κ is a limit cardinal,
- (iv) κ is a strong limit cardinal.

As consequence, we get

Proposition 3.11. If $M \subseteq N$ is transitive model of ZFC and $\kappa \in M$ and $\mathcal{N} \models \kappa$ is (weakly) inaccessible, then $\mathcal{M} \models \kappa$ is (weakly) inaccessible.

Axiom. GCH: Generalised Continuum Hypothesis

$$\forall \alpha. 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

Remark. Note that if κ is a limit cardinal $\iff \forall \alpha. \aleph_\alpha < \kappa \rightarrow \aleph_{\alpha+1} < \kappa$ and κ is a strong limit cardinal $\iff \forall \alpha. \aleph_\alpha < \kappa \rightarrow 2^{\aleph_\alpha} < \kappa$.

Then if GCH is consistent, then weakly inaccessible cardinals are strongly inaccessible.

Definition 3.12 (WIC). Let

$$\text{WIC} := \exists \kappa (\kappa \text{ is weakly inaccessible}).$$

We're going to argue, if ZFC is consistent, then $\text{ZFC} \not\models \text{WIC}$.

Definition 3.13 (Inner Model). $M \subseteq N$, $\mathcal{M}$ transitive in $\mathcal{N}$, both models of ZFC. We say $\mathcal{M}$ is an **inner model** of $\mathcal{N}$ if $\text{Ord} \cap M = \text{Ord} \cap N$.

Definition 3.14 (Definable). $\mathcal{M}$ is **definable** in $\mathcal{N}$ if there is a formula Φ such that

$$x \in M \iff \mathcal{N} \models \Phi(x).$$

Proposition 3.15. If $\mathcal{M}$ is definable in $\mathcal{N}$, then the relation $\mathcal{M} \models \varphi$ is first-order expressible in $\mathcal{N}$.

Proof. We introduce here the method of **relativisation**. Translate φ into a relativist φ^M such that

$$\mathcal{N} \models \varphi^M \iff \mathcal{M} \models \varphi.$$

We do this by recursion on the formulas:

- (i) $\varphi^M := \varphi$ if φ is atomic,
- (ii) $(\varphi \wedge \psi)^M := \varphi^M \wedge \psi^M$,
- (iii) $(\neg \varphi)^M := \neg(\varphi^M)$,
- (iv) $(\exists x \varphi)^M := \exists x (\Phi(x) \wedge \varphi)$.

By an easy induction using the definition, we get

$$\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi^M.$$

□

Theorem 3.16 (Gödel '38). If $(N, \in) \models \text{ZFC}$, then there is a definable inner model $\mathcal{M}$ such that

$$(M, \in) \models \text{ZFC} + \text{GCH}.$$

Proof. See [Gö38].

□

Corollary 3.17. $\text{Cons}(\text{ZFC}) \implies \text{Cons}(\text{ZFC} + \text{GCH})$.

Remark. There is a uniform formula Φ defining $\mathcal{M}$ in every model of ZFC, called **Gödel's Constructible Universe**, denoted L .

Remark. The theorem can be strengthened as

$$(N, \in) \models \text{ZF} \implies (M, \in) \models \text{ZFC} + \text{GCH},$$

giving consistency of choice.

Corollary 3.18. If ZFC is consistent, then $\text{ZFC} \not\vdash \text{WIC}$.

Proof. Let $(N, \in) \models \text{ZFC}$. By Gödel 3.16, there is a definable $M \subseteq N$ such that

$$(M, \in) \models \text{ZFC} + \text{GCH}.$$

Towards a contradiction, assume $\text{ZFC} \vdash \text{WIC}$, so

$$\begin{aligned} (M, \in) &\models \text{ZFC} + \text{GCH} + \text{WIC} \\ \implies (M, \in) &\models \text{ZFC} + \text{IC}. \end{aligned}$$

We now apply Hausdorff's proof 2.11 to show that if κ is inaccessible in M , then

$$V_\kappa^M := \{x \in M; \mathcal{M} \models \rho(x) < \kappa\},$$

is a set in N , by Proposition 3.15, since $\mathcal{M} \models \rho(x) < \kappa$ can be expressed as a first order formula, so it can be separated. Hausdorff's proof 2.11 gives that

$$\mathcal{N} \models (V_\kappa^M \models \text{ZFC}),$$

so we argued that $\text{ZFC} \vdash \text{Cons}(\text{ZFC})$. This contradicts incompleteness. $\square$

3.2 A Non-Transitive Model of ZFC

Aim. To construct a non-transitive $M \subseteq N$ such that $\mathcal{M} \models \text{ZFC}$.

Remark. This is an improvement of downward Lowenheim-Skolem used to obtain countable elementary substructure of $\mathcal{N}$.

Idea. The proof technique (using TVT) is precisely the one used in our proof of the existence of worldly cardinals (Theorem 2.23). Do the same proof but only include the witnesses.

Construction 3.19. We construct $\mathcal{M}$ as follows: we start with $M_0 := \emptyset$. If M_i is defined (and countable), then consider all $\vec{x} \in M_i^{<\omega}$, formulas φ . If

$$\mathcal{N} \models \exists x \varphi(x, \vec{x}),$$

we take $w(\varphi, \vec{x})$ to be any witness, picking it with AC if necessary. If there is no such witness, we take $w(\varphi, \vec{x}) := \emptyset$, and then define

$$M_{i+1} := M_i \cup \{w(\varphi, \vec{x}); \varphi \text{ formula}, \vec{x} \in M_i^{<\omega}\}.$$

By induction, M_i is countable and so is $M := \bigcup_{i \in \mathbb{N}} M_i$, as countable union of countable sets, by AC.

The TVT argument in 2.23 shows $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ ($\mathcal{M} \preceq \mathcal{N}$). In particular $\mathcal{M} \models \text{ZFC}$. If $N = V_\kappa$ where κ is inaccessible or even worldly, then $M \neq N$.

Remark. Scott's trick, i.e. picking witnesses of minimal rank, is needed to address the situation of formulae a proper class of witnesses. For example, one such is

$$\varphi(x) := \exists y. y \in x$$

Claim. $\mathcal{M}$ cannot be transitive in $\mathcal{N}$.

Proof. Consider the formula:

$$\varphi(x) = "x \text{ is the smallest uncountable ordinal}."$$

The formula $\exists x \varphi(x)$ has precisely one witness in N , namely $x = \aleph_1^N$. So $\aleph_1 \in M_1 \subseteq M$. By countability of M , we have $\aleph_1 \cap M$ is countable, so $\aleph_1 \notin M$. Hence $\mathcal{M}$ is not transitive. $\square$

Remark. $\mathcal{M}$ looks like $\mathcal{N}$ but the witnesses are incredibly sparse. But being an elementary substructure, it is not only a model of ZFC, it is indistinguishable for $\mathcal{N}$!

Remark. Modern large cardinals axioms built on modern understanding of model theory. Increasing the strength is the same as increasing the set theory that goes on in whatever model implies it exists. But it's not the historical way to see large cardinals.

Remark. The hierarchy is almost linear, and that quite a puzzling phenomenon. If you design your large cardinals from the point of view of model theory, this is expected, so linearity in the highest ranges is not surprising at all. However, it was initially quite a big surprise when it was proven that large cardinal axioms were comparable.

Remark. The precise definition of consistency strength is very non-linear. But it happens to be for large cardinals. This is very unexpected as there is no strict definition of large cardinals. We're somehow able to come up with something that is a large cardinal axiom that no-one thought about it before and agree that is a large cardinal.

Remark. For example, see [GS18] for a new group of large cardinals around measurable. They're a group of somewhat model theoretic cardinals that no-one had thought about it before.

Question. How to prove that being an ordinal is not a Δ_0 formula?

Answer. In a pretty good place with the model of ZFC we constructed. But the model we constructed is countable, so not all cardinals are *real*.

To get counter-examples, need GCH to fail badly, which might require forcing. By Gödel's theorem, this requires extra assumptions, as any model of ZFC could be a model of GCH.

Lecture 6

4 Towards Larger Cardinals

Aim. We want to move towards the following larger large cardinals notations:

- (i) Strongly compact,
- (ii) Measurable cardinals,
- (iii) Weakly compact cardinals.

and show the (non-)implications.

4.1 Measurable Cardinals

Question (Measure Problem). Is there a function

$$\mu : \mathcal{P}([0, 1]) \rightarrow [0, 1],$$

such that

- (i) $\mu([0, 1]) = 1$,
- (ii) $\mu(\emptyset) = 0$,
- (iii) $\forall x \in [0, 1]. \mu(A + x) = \mu(A)$,
- (iv) If $\{A_i \subseteq [0, 1]; i \in \mathbb{N}\}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

Answer (Vitaly 1905). If AC holds, there is a non-measurable set. More precisely, if there is a basis for $\mathbb{R}$ as a $\mathbb{Q}$ -vector space, then there is a non-measurable set.

Answer (In Measure Theory). Give up on $\text{dom}(\mu) = \mathcal{P}([0, 1])$ and work with appropriate σ -algebras instead.

Question. Under what circumstances can you get something like this? Can you weaken the axiom of choice? Is there something special about $[0, 1]$? What about different underlying sets?

Question (Banach). Drop translation invariance, the only geometric property.

Answer. This introduces trivial solutions, like

$$\mu(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

This is a solution to the measure problem without translation.

Definition 4.1 (Non-Trivial Measure/Function). We say μ is **non-trivial** if for all $x \in \text{dom}(\mu)$, $\mu(\{x\}) = 0$.

Remark. By σ -additivity, all countable sets get measure 0.

With no more ties to geometry of $[0, 1]$, we can change arbitrary sets:

Definition 4.2 (Ulam Measure). An **Ulam measure on X** is a function

$$\mu : \mathcal{P}(X) \longrightarrow \{0, 1\},$$

that is non-trivial, has $\mu(X) = 1$, $\mu(\emptyset) = 0$, and is σ -additive.

Remark. If $A \cap B = \emptyset$, then at most one of A and B can have measure 1.

Definition 4.3 (Ulam Measurable). A cardinal κ is called **Ulam measurable** if there is an Ulam measure on κ .

Recall that σ -additivity was necessary for the measure theoretic interest. When working in generality, there is no reason to talk about countable unions only:

Definition 4.4 (κ -Additivity). We say that an Ulam measure μ on κ is **κ -additive** if for all $\lambda < \kappa$ and families $\{A_\alpha; \alpha < \lambda\}$ such that the A_α s are pairwise disjoint, we have

$$\mu\left(\bigcup_{\alpha < \lambda} A_\alpha\right) = \sum_{\alpha < \lambda} \mu(A_\alpha).$$

Note that the sum over potentially very large cardinals is well-defined, since for pairwise disjoint sets, $\mu(\bigcup \dots) = 1$ iff there is $\alpha < \lambda$ with $\mu(A_\alpha) = 1$.

Definition 4.5 (Measurable Cardinal). A cardinal κ is **measurable** if there is a κ -additive Ulam measure on it.

Proposition 4.6. If $A \subseteq \kappa$ has cardinality $|A| < \kappa$, then $\mu(A) = 0$.

Proof. Write $A = \{a_\alpha; \alpha < |A|\}$. Then by κ -additivity, we have

$$\mu(A) = \mu\left(\bigcup_{\alpha < |A|} \{a_\alpha\}\right) = \sum_{\alpha < |A|} \mu(\{a_\alpha\}) = 0,$$

by non-triviality. □

4.2 Filters and Ultrafilters

Definition 4.7 (Filter). A family $F \subseteq \mathcal{P}(\kappa)$ is called a **filter on κ** if

- (i) $\kappa \in F$,
- (ii) $\emptyset \notin F$,
- (iii) $A, B \in F \implies A \cap B \in F$,
- (iv) $A \in F, A \subseteq B \implies B \in F$,

Note that condition (iii) is precisely $\aleph_0$ -completeness.

Definition 4.8. A filter is

- (i) **Non-trivial** if for all $\alpha \in \kappa$, $\{\alpha\} \notin F$,
- (ii) **Free** if $\bigcap_{X \in \mathcal{F}} X = \emptyset$,
- (iii) **Fixed** if $\bigcap_{X \in \mathcal{F}} X \neq \emptyset$,
- (iv) **λ -Complete** if for all $\mu < \lambda$ and $\{A_\alpha; \alpha < \mu\} \subseteq F$, we have

$$\bigcap_{\alpha < \mu} A_\alpha \in F.$$

Definition 4.9 (Ultrafilter). A filter is called an **ultrafilter** if for all $A \subseteq \kappa$, either $A \in F$ or $\kappa \setminus A \in F$.

Remark. Not a coincidence that this looks like a σ -algebra. They are similar notions of closure of certain families of subsets.

Proposition 4.10. An ultrafilter is free iff it is non-trivial.

Warning. The same does not hold for filters.

Remark. κ -complete non-trivial ultrafilters and κ -additive Ulam measures are corresponding notions:

$$\begin{aligned} \{U \text{ ultrafilter}\} &\longleftrightarrow \{\mu(A) = 1 \iff A \in U\} \\ \{\mu \text{ measure}\} &\longleftrightarrow \{A \in U \iff \mu(A) = 1\} \end{aligned}$$

Some set algebra is needed to check that additivity and completeness translate into each other in this particular case. For example, if U is an ultrafilter, k -completeness is equivalent to

$$(\forall A_\alpha \in \{A_\alpha \subset \kappa; \alpha < \lambda\}. A_\alpha \notin U) \implies \bigcup_{\alpha < \lambda} A_\alpha \notin U. \quad (*)$$

Theorem 4.11 (Measurable Regular). If κ is measurable, κ is regular.

Proof. Let U be the κ -complete ultrafilter on κ . By $(*)$, sets of size $< \kappa$ cannot be in U . If $\kappa = \bigcup C$ cofinal and $|C| < \kappa$, then each $\alpha \in C$ has size $< \kappa$, so $\alpha \notin U$, but then κ is a $|C|$ -union of things not in U , so by κ -completeness $\kappa \notin U$. Contradiction. $\square$

Theorem 4.12. If κ is measurable, then κ is a strong limit.

Proof. If κ is not a strong limit, then there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$, i.e. there is an injection $\kappa \hookrightarrow \mathcal{P}(\lambda)$. Therefore there is an injection

$$f : \kappa \longrightarrow \{g; g : \lambda \rightarrow 2\},$$

write $f_\alpha : \lambda \rightarrow 2$ for $f(\alpha)$.

Fix an ordinal $\xi \in \lambda$, consider $\{\alpha; f_\alpha(\xi) = 0\} \cup \{\alpha; f_\alpha(\xi) = 1\}$. Precisely one of these sets is in the ultrafilter U . Let b_ξ be either 0 or 1 depending on which set is in U and define $A_\xi = \{\alpha; f_\alpha(\xi) = b_\xi\} \in U$. Since $A_\xi \in U$ for each $\xi \in \lambda$ and U is κ -complete, we have

$$\begin{aligned} A &:= \bigcap_{\xi < \lambda} A_\xi \\ &= \bigcap_{\xi < \lambda} \{\alpha; f_\alpha(\xi) = b_\xi\} \\ &= \{\alpha; \forall \xi < \lambda. f_\alpha(\xi) = b_\xi\} \in U. \end{aligned}$$

But $|A| \leq 1$, so $A \notin U$. Contradiction. $\square$

Corollary 4.13. All measurable cardinals are inaccessible.

Remark. Measurable cardinals are the protagonist of this course and will feature prominently in the second half of this course. Their existence has a huge impact on the structure of the set theoretic universe. For now, all we know is that they are large cardinals.

5 Compactness

Remark. In FOL, **compactness** is the statement:

"If Φ is finitely satisfiable, then Φ is satisfiable."

Remark. Compactness is a blessing and a curse. It is very powerful but also it is the source of many limitations of first order logic.

Example. We cannot characterise finite structures or countable structures.

Idea (Tarski's). Expand FOL to allow infinite formulae. We would like to say

$$x = 0 \vee x = 1 \vee \dots,$$

which is allowed in infinitary logic.

Definition 5.1 ($\mathcal{L}_{\kappa,\lambda}$ -Language). Let κ, λ be cardinals, and define $\mathcal{L}_{\kappa,\lambda}$ -languages by

- (i) Allowing arbitrary families of variables, even proper classes of them,
- (ii) S non-logical symbol set, which could be infinite/uncountable,
- (iii) $\vee, \wedge, \neg, \implies, \exists, \forall$ with usual semantics,
- (iv) $\bigwedge_{\alpha < \mu} \varphi_\alpha$ and $\bigvee_{\alpha < \mu} \varphi_\alpha$, for $\mu < \kappa$,
- (v) $\exists^\mu \vec{x} \varphi$ and $\forall^\mu \vec{x} \varphi$, where $\vec{x}$ is a μ -sequence of variables and $\mu < \lambda$.

Question. Is there an analogue of compactness for infinitary logic?

Lecture 7

Aim. We wish to define strongly and weakly compact and prove some implications in **local strength**.

Remark. The semantics of $\mathcal{L}_{\kappa,\lambda}$ -languages can be found in [Jec03]. The non-logical symbols are S , $\bigvee_{\alpha < \mu} \varphi_\alpha$, $\bigwedge_{\alpha < \mu} \varphi_\alpha$ for $\mu < \kappa$ and $\exists^\nu \vec{x} \varphi, \forall^\nu \vec{x} \varphi$ for $\nu < \lambda$, and the language is constructed as follows:

- (i) Terms are constructed from S exactly as in first order logic,
- (ii) Atomic formulas are define exactly as in first order logic,
- (iii) $\mathcal{M} \models \bigwedge_{\alpha < \mu} \varphi_\alpha \iff$ for all $\alpha < \mu$, $\mathcal{M} \models \varphi_\alpha$, and similarly for $\bigvee_{\alpha < \mu}$.
- (iv) $\mathcal{M} \models \exists^\nu \vec{x} \varphi \iff \exists a : \nu \rightarrow M$ such that

$$\mathcal{M}[\vec{x}(\alpha) \mapsto a(\alpha)] \models \varphi,$$

read the variable $\vec{x}(\alpha)$ is interpreted by $a(\alpha)$.

Remark. Infinitary logic can express multiple features that are beyond FOL.

Example (Characterization of Finite Structures). Consider

$$\varphi_{\text{fin}} : \iff \forall^\omega \vec{x} \bigvee_{i \neq j} x_i = x_j,$$

an $\mathcal{L}_{\omega_1, \omega_1}$ -formula. Clearly, $\mathcal{M} \models \varphi_{\text{fin}}$ iff $\mathcal{M}$ is a finite structure, which cannot be expressed in first order logic.

Remark. This immediately shows that the usual formulation of compactness for FOL cannot be true here.

Example (Countable Structures). Let c_n be a constant symbol for $n \in \mathbb{N}$, then

$$\forall x \bigvee_{n \in \mathbb{N}} x = c_n,$$

describes models that are at most countable. Furthermore, the model is in bijection with $\mathbb{N}$ if we add

$$\bigwedge_{n \neq m} c_n \neq c_m.$$

Definition 5.2 (μ -Satisfiable). If Φ is a set of formulae in an $\mathcal{L}_{\kappa, \lambda}$ -language, we say that Φ is μ -**satisfiable** if for each subset $\Phi_0 \subseteq \Phi$ such that $|\Phi_0| < \mu$, Φ_0 is satisfiable.

Remark. Finite satisfiable $\iff \aleph_0$ -satisfiable.

Definition 5.3 (Strongly compact). A cardinal κ is called **strongly compact** if for every $\mathcal{L}_{\kappa, \kappa}$ -language L and every set Φ of L -sentences, if Φ is κ -satisfiable, then it is satisfiable.

Definition 5.4 (Weakly compact). A cardinal κ is called **weakly compact** if for every $\mathcal{L}_{\kappa, \kappa}$ -language L with set non-logical symbols $|S| \leq \kappa$ and every set Φ of L -sentences, if Φ is κ -satisfiable, then it is satisfiable.

Remark. This might be the only definition in the course in which $\leq$ shows up.

A priori, we have no reason to believe these are large cardinal notions.

Aim. Want to prove compactness notions imply inaccessibility.

Theorem 5.5. If κ is weakly compact, then κ is regular.

Proof. Suppose $\kappa = \bigcup X$ where $|X| < \kappa$. Take an $\mathcal{L}_{\kappa, \kappa}$ -language with constant symbols c_α , for $\alpha < \kappa$, plus an extra c . Consider the FOL formulae $c_\alpha \neq c$ for $\alpha < \kappa$, and write the $\mathcal{L}_{\kappa, \kappa}$ -formula

$$\psi := \bigvee_{\beta \in X} \bigvee_{\alpha < \beta} c = c_\alpha.$$

Note that the disjunctions are of size $|X|, \beta < \kappa$, as required by the $\mathcal{L}_{\kappa, \kappa}$ -language.

Let $\Phi := \{\psi\} \cup \{c_\alpha \neq c; \alpha < \kappa\}$. Clearly, Φ is κ -satisfiable, but not satisfiable. Contradiction. $\square$

Theorem 5.6 (Keisler-Tarski). If κ is strongly compact and $\mathcal{F}$ is a κ -complete filter on an arbitrary set X , then $\mathcal{F}$ can be extended to a κ -complete ultrafilter on X .

Proof. Fix X and $\mathcal{F}$ a κ -complete filter on X . By AC, we can take without loss of generality X to be an ordinal. We now add a constant c_A for every $A \subseteq X$, which gives $2^{|X|}$ non-logical symbols, plus an extra constant c .

Let L be the $\mathcal{L}_{\kappa,\kappa}$ -language with $\in$ and non-logical symbols c_A , L^* the $\mathcal{L}_{\kappa,\kappa}$ -language with $\in, c_A, c$,

$$\mathcal{M} := (\mathcal{P}(X), \in, \{A; A \subseteq X\}),$$

and $\Phi := \text{Th}_{\mathcal{L}_{\kappa,\kappa}}(\mathcal{M})$.

Note that

$$\alpha \text{ is an ordinal} \iff \mathcal{M} \models \alpha \text{ is a transitive set totally ordered by } \in,$$

so if $A \subseteq X$, then

$$\begin{aligned} \mathcal{M} &\models \forall x (x \text{ is an ordinal} \wedge x \neq X \rightarrow (x \in c_A \vee x \in c_{X \setminus A})), \\ \mathcal{M} &\models \forall x (x \in c_A \rightarrow x \text{ is an ordinal}). \end{aligned}$$

Consider now $\Phi^* := \Phi \cup \{c \in c_A; A \in \mathcal{F}\}$. We have Φ^* is κ -satisfiable, as if $\Phi_0 \subseteq \Phi^*$ has size $< \kappa$, then the intersection of all A such that $c \in c_A$ occurs in Φ_0 , so any element in the intersection can serve as the interpretation of c . This shows that Φ^* is satisfiable.

Let $\mathcal{M}^*$ be a model of Φ^* and define

$$U := \{A \subseteq X; \mathcal{M}^* \models c \in c_A\}.$$

By the above, $\mathcal{M}^* \models c$ is an ordinal.

We now want to show that U is a κ -complete ultrafilter extending $\mathcal{F}$.

- (i) Since $c \in c_A$ for $A \subseteq \mathcal{F}$ is in Φ^* , we immediately get that $\mathcal{F} \subseteq U$.
- (ii) Since

$$\begin{aligned} \mathcal{M} &\models \forall x (x \text{ is an ordinal} \rightarrow (x \in c_A \vee x \in c_{X \setminus A})), \\ \mathcal{M}^* &\models c \text{ is an ordinal,} \end{aligned}$$

we have that U is an ultrafilter.

- (iii) For κ -completeness, let A_α for $\alpha < \lambda$ be such that

$$\mathcal{M}^* \models c \in c_{A_\alpha},$$

and define $A := \bigcap_{\alpha < \lambda} A_\alpha$. We want to show $\mathcal{M}^* \models c \in c_A$. We can describe A in $\mathcal{L}_{\kappa,\kappa}$ as

$$\begin{aligned} \mathcal{M} &\models \forall x (x \in c_A \leftrightarrow \bigwedge_{\alpha < \lambda} x \in c_{A_\alpha}) \\ \implies \mathcal{M}^* &\models c \in c_A \leftrightarrow \bigwedge_{\alpha < \lambda} c \in c_{A_\alpha}. \end{aligned}$$

Since we assumed the RHS to be true, $\mathcal{M}^* \models c \in c_A$, hence $A \in U$.

□

Idea. The takeaways from Theorem 5.6 are as follows:

- (i) We want to take X to be an ordinal because it $\in$ on $\mathcal{P}(X)$ meaningful.
- (ii) The strength of Theorem 5.6 is that we're preserving κ -completeness.
- (iii) κ -Completeness is encoded as a big conjunction over non-logical symbols representing subsets of X .

Corollary 5.7. Any strongly compact cardinal is measurable.

Proof. We need to show existence of a κ -complete filter on κ such that no extension to an ultrafilter can be trivial. Hence take

$$\mathcal{F} := \{A \subseteq \kappa : |\kappa \setminus A| < \kappa\}.$$

By Theorem 5.5, κ is regular, and so $\mathcal{F}$ is κ -complete, otherwise could write κ as a small union of *things*. Therefore by Keisler-Tarski 5.6, it can be extended to a κ -complete ultrafilter, which cannot be trivial. $\square$

Lecture 8

Theorem 5.8. Weakly compact implies inaccessible.

Proof. Already know *regular*, need to show *strong limit*. Towards a contradiction, assume that there is some $\lambda < \kappa$ with $2^\lambda \geq \kappa$. We design an $\mathcal{L}_{\kappa, \kappa}$ -language with constant symbols $c_\alpha, c_\alpha^0, c_\alpha^1$ for $\alpha < \kappa$, which is allowed by weak compactness. Let Φ be given by

$$\Phi := \left\{ \bigwedge_{\alpha < \lambda} c_\alpha^0 \neq c_\alpha^1, \bigwedge_{\alpha < \lambda} c_\alpha = c_\alpha^0 \vee c_\alpha = c_\alpha^1 \right\}.$$

and set Φ^* to be

$$\Phi^* := \Phi \cup \left\{ \bigvee_{\alpha < \lambda} c_\alpha \neq c_\alpha^{f(\alpha)}; f : \lambda \rightarrow 2 \right\}.$$

If $\mathcal{M} \models \Phi$, we can define a function $g : \lambda \rightarrow 2$ by

$$g(\alpha) := \begin{cases} 0 & \text{if } \mathcal{M} \models c_\alpha = c_\alpha^0 \\ 1 & \text{if } \mathcal{M} \models c_\alpha = c_\alpha^1 \end{cases}$$

Thus, Φ^* is inconsistent since the additional formula says that no such function can exist. To conclude our contradiction, we need to show that Φ^* is κ -satisfiable. Take $\Phi_0 \subseteq \Phi^*$ such that $|\Phi_0| < \kappa \leq 2^\lambda$. Therefore, there is some $f : \lambda \rightarrow 2$ such that $\varphi_f \in \Phi_0$. But if we now interpret c_α by f , that will satisfy Φ_0 . $\square$

5.1 Method of Ultraproducts

Let U be an ultrafilter on κ and $\vec{\mathcal{M}} = (\mathcal{M}_\alpha; \alpha < \kappa)$ a sequence of first order structures. Consider the choice functions for $\vec{\mathcal{M}}$:

$$f : \kappa \rightarrow \bigcup_{\alpha < \kappa} M_\alpha,$$

such that $f(\alpha) \in M_\alpha$. Define the equivalence relation on the set of choice functions

$$f \sim_U g \iff \{\alpha; f(\alpha) = g(\alpha)\} \in U.$$

Definition 5.9 (Ultraproduct). We write $\text{Ult}(\vec{\mathcal{M}}, U)$ for the quotient of the set of choice functions modulo $\sim_U$, and call it the **ultraproduct** of $\mathcal{M}_\alpha$, writing $[f]$ for the equivalence class of f .

Exercise. Check that the obvious definitions, e.g. if R is a binary relation symbol, then $[f]R[g] \iff \{\alpha < \kappa; \mathcal{M}_\alpha \models f(\alpha)Rg(\alpha)\} \in U$, are well-defined and give a structure of the right type.

Theorem 5.10 (Los). If φ is a formula in n variables, then

$$\text{Ult}(\vec{\mathcal{M}}, U) \models \varphi([f_1], \dots, [f_n]) \iff \{\alpha < \kappa; \mathcal{M}_\alpha \models \varphi(f_1(\alpha), \dots, f_n(\alpha))\} \in U.$$

Proof. By induction on complexity of φ . See discussion in [Kei10] or Logic and Computability. $\square$

Remark. If φ is a sentence, Los gives us

$$\text{Ult}(\vec{\mathcal{M}}, U) \models \varphi \iff \{\alpha < \kappa; \mathcal{M}_\alpha \models \varphi\} \in U,$$

so if all $\mathcal{M}_\alpha \models \varphi$, then so is $\text{Ult}(\vec{\mathcal{M}}, U)$.

Definition 5.11 (Ultrapower). We call the ultraproduct $\text{Ult}(\vec{\mathcal{M}}, U)$ an **ultrapower**, denoted $\text{Ult}(\mathcal{M}, U)$, if $\vec{\mathcal{M}}$ is constant, i.e. if for all α , $\mathcal{M}_\alpha = \mathcal{M}$.

Definition 5.12 (Ultrapower Embedding). We call the map

$$\begin{aligned} j : M &\longrightarrow \text{Ult}(\mathcal{M}, U) \\ m &\longmapsto [c_m : \alpha \longmapsto m], \end{aligned}$$

the **ultrapower embedding**.

Corollary 5.13. The ultrapower embedding is an elementary embedding.

Proof. Immediate from Los' Theorem 5.10, otherwise we show for $m_1, \dots, m_n \in M$

$$\begin{aligned} \mathcal{M} \models \varphi(m_1, \dots, m_n) &\iff \text{Ult}(\mathcal{M}, U) \models \varphi(j_U(m_1), \dots, j_U(m_n)) \\ &\iff \text{Ult}(\mathcal{M}, U) \models \varphi([c_{m_1}], \dots, [c_{m_n}]) \\ &\iff \{\alpha < \kappa; \mathcal{M} \models \varphi(c_{m_1}, \dots, c_{m_n})\} \in U \\ &\iff \{\alpha < \kappa; \mathcal{M} \models \varphi(m_1, \dots, m_n)\} \in U \\ &\iff \mathcal{M} \models \varphi(m_1, \dots, m_n), \end{aligned}$$

where the last step follows from independence from α . We can think of $\mathcal{M}$ as an elementary substructure of $\text{Ult}(\mathcal{M}, U)$ by identifying $\mathcal{M}$ with $\{j_U(m); m \in M\}$. $\square$

We now want to use this to prove Theorem 5.15.

Lemma 5.14. If L is an $\mathcal{L}_{\kappa,\kappa}$ -language with $\leq \kappa$ non-logical symbols for κ inaccessible, then $|L| \leq \kappa$.

Proof. Clearly, we have at most κ -many atomic formulae. Inductively, we only need to show that the construction steps for $\mathcal{L}_{\kappa,\kappa}$ -languages preserves "cardinality $\leq \kappa$ ", i.e. if X is the set of formulas with $|X| \leq \kappa$, then the closure of X under the $\mathcal{L}_{\kappa,\kappa}$ -constructions has card. $\leq \kappa$. In this case, closing under $\bigwedge_{\alpha < \lambda} \varphi_\alpha$ will give cardinality $\kappa^{<\kappa} = \bigcup_{\lambda < \kappa} \kappa^\lambda$. Since κ is inaccessible and in particular regular, cardinal algebra gives us $\kappa^{<\kappa} = \kappa$. Similarly, $\exists^\lambda \vec{x} \varphi$ will give rise to $\kappa^{<\kappa}$ many formulae, so $\leq \kappa$. $\square$

Theorem 5.15. All measurables are weakly compact.

Proof. Suppose κ is measurable. Then we know that κ is inaccessible. If Φ is any set of formulae in an $\mathcal{L}_{\kappa,\kappa}$ -language with $\leq \kappa$ non-logical symbols, we can write it as

$$\Phi = \{\varphi_\alpha; \alpha < \kappa\},$$

by our Lemma 5.14. We now need to show that if Φ is κ -satisfiable, then it is satisfiable. Define

$$\Phi_\lambda := \{\varphi_\alpha; \alpha < \lambda\}.$$

By assumption, Φ_λ is satisfiable. Thus let $\mathcal{M}_\lambda \models \Phi_\lambda$, and define now $\mathcal{M} := \text{Ult}(\vec{\mathcal{M}}, U)$, where U is a non-trivial κ -complete ultrafilter on κ , which exists by measurability. Recall that $|A| < \kappa$, then $A \notin U$ by κ -completeness and non-triviality.

We now claim that $\mathcal{M} \models \Phi$. Let φ_α be arbitrary and consider

$$\{\lambda < \kappa; \mathcal{M}_\lambda \models \varphi_\alpha\} \supseteq \{\lambda < \kappa; \alpha < \lambda\} \in U$$

So $\{\lambda < \kappa; \mathcal{M}_\lambda \models \varphi_\alpha\} \in U \iff \text{Ult}(\vec{\mathcal{M}}, U) \models \varphi_\alpha$ by Los' Theorem 5.10. $\square$

Idea. Proof breakdown:

- (i) Use inaccessible to get $|L| < \kappa$ for any $\mathcal{L}_{\kappa,\kappa}$ with $\leq \kappa$ -many non-logical symbols,
- (ii) Then κ -enumerate any set of $\mathcal{L}_{\kappa,\kappa}$ -formulae,
- (iii) Assume κ -satisfiability and take ultrapower of models by U on κ ,
- (iv) Arbitrary formula must hold by κ -completeness, done by Los' 5.10.

This concludes the final implication.

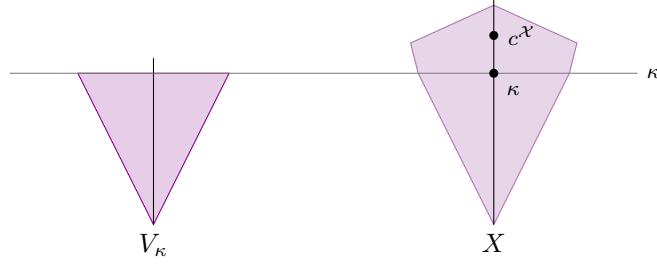
Lecture 9

5.2 Method of Reflection

Idea. Want to prove inaccessibility does not imply weak compactness, by moral converse of Theorem 2.16 called the **reflection technique**:

Definition 5.16 (KEP). We say κ has the **Keisler Extension Property** if

$$\exists \text{ transitive } X \supset V_\kappa. (V_\kappa, \in) \preceq (X, \in).$$

Figure 1: Construction of $c^X \in X$ given κ .

Idea. KEP is dual to the properties of inaccessibles and worldlies.

Remark. Being weakly compact is actually equivalent to having a stronger version of the KEP, but we won't show it.

Need the following result from Logic and Set Theory:

Theorem 5.17 (Mostowski's Collapsing). Let R be a relation on a set b that is well-founded and extensional. Then there exists a transitive set a , and a bijection $f : a \rightarrow b$ such that

$$(\forall x, y \in a)(xRy \iff f(x) \in f(y)).$$

Moreover, b and f are unique.

Theorem 5.18. If κ is weakly compact, then κ has the KEP.

Proof. Observe that κ is inaccessible, thus by Hausdorff's Theorem 2.11, we have that $|V_\kappa| = \kappa$ and if $x \in V_\kappa$, then $|x| < \kappa$.

We define an $\mathcal{L}_{\kappa, \kappa}$ -language $\mathcal{L}$ with additional constant symbols c_x for each $x \in V_\kappa$ and $\mathcal{L}^*$ with an additional constant symbol c . By Hausdorff, this language has κ -many non-logical symbols. Define the $\mathcal{L}$ -structure and the theories

$$\begin{aligned} \mathcal{V} &:= (V_\kappa, \in, \{x; x \in V_\kappa\}) \\ \Phi &:= \text{Th}_{\mathcal{L}}(\mathcal{V}) \\ \Phi^* &:= \Phi \cup \{c \neq c_x; x \in V_\kappa\} \cup \{c \text{ is an ordinal}\} \end{aligned}$$

Observe that the formula

$$\psi := \forall^\omega \vec{x} \bigvee_{i \in \mathbb{N}} x_{i+1} \notin x_i,$$

describes well-foundedness and is an $\mathcal{L}_{\omega_1, \omega_1}$ -formula. Clearly $\mathcal{V} \models \psi$, so $\psi \in \Phi$ and hence any model of $\Phi^* \supseteq \Phi$ will be well-founded. Also, since κ is inaccessible, $\text{ZFC} \subseteq \Phi$, and in particular the axiom of extensionality, and thus any model of Φ^* will be extensional.

Clearly, Φ^* is κ -satisfiable: if $\Phi_0 \subseteq \Phi^*$ with $|\Phi_0| < \kappa$, then there is an ordinal $\alpha < \kappa$ such that $c \neq c_\alpha \notin \Phi_0$ and we can interpret c by α . By weak compactness, Φ^* is satisfiable.

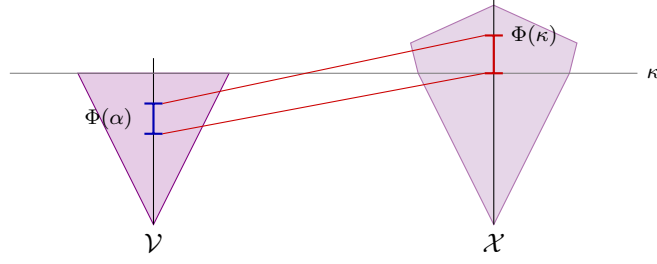


Figure 2: Reflection.

Let $\mathcal{M} \models \Phi^*$. By the above, $\mathcal{M}$ is well-founded and extensional. Thus by Mostowski's Collapse Theorem 5.17, there is a transitive X such that

$$(X, \in) \cong (M, E).$$

Write $\mathcal{X} := (X, \in, \dots)$. First show that $V_\kappa \subseteq X$:

$$\mathcal{V} \models \forall z (z \in c_x \leftrightarrow \bigvee_{y \in x} z \in c_y),$$

since $|x| < \kappa$. So, this formula is in Φ , and hence true in $\mathcal{M}$ and in $(X, \in)$. Thus by a simple induction on rank, we get $c_x^{\mathcal{X}} = x$. Since $\mathcal{X} \models \Phi^*$, we get that $c^{\mathcal{X}}$ is an ordinal, and, moreover, $c^{\mathcal{X}}$ is bigger than all $\alpha < \kappa$. Thus $X \supsetneq V_\kappa$. Therefore $x \mapsto c_x^{\mathcal{X}} = x$ is the identity. By construction, this is elementary. $\square$

We are in the situation described in Figure 1. Note that $k \in X$.

Theorem 5.19. If κ is weakly compact, then there is $\lambda < \kappa$ inaccessible.

Proof. By KEP, we can find $X \supsetneq V_\kappa$ such that

$$(V_\kappa, \in) \preceq (X, \in).$$

Since $\kappa \in X$ and inaccessibility is downwards absolute for transitive sets,

$$\begin{aligned} \mathcal{X} \models \kappa \text{ is inaccessible} \\ \implies \mathcal{X} \models \text{IC}. \end{aligned}$$

However, $(V_\kappa, \in) \equiv (X, \in)$, so $V_\kappa \models \text{IC}$. Hence there is $\lambda < \kappa$ such that

$$V_\kappa \models \lambda \text{ is inaccessible}.$$

Thus, by ES#1, λ is inaccessible. $\square$

Corollary 5.20. The least inaccessible cannot be weakly compact.

Remark. The property depicted in Figure 2 is called **reflection**: a property that holds in the taller universe $\mathcal{X}$ is reflected downwards in a smaller universe.

A step-by-step analysis of this argument:

- (i) Suppose $\Phi(\kappa)$ holds,
- (ii) Argue that $\mathcal{X} \models \Phi(\kappa)$,
- (iii) Using elementarity to get $\mathcal{V} \models \exists \alpha. \Phi(\alpha)$,

(iv) Argue that there is $\alpha < \kappa$ such that $\Phi(\alpha)$.

6 Logical Hierarchies

6.1 Two Naive Notions

Question. How do we measure the logical strength of theories, or more specifically, of large cardinal axioms?

Answer (Naive). Use logical consequence: if Φ is a set of sentences, then we define

$$C_\Phi := \{\psi; \Phi \vdash \psi\}.$$

Let

$$\Phi \leq_0 \Psi : \Longleftrightarrow C_\Phi \subseteq C_\Psi.$$

In particular, we have

$$\text{ZFC} \leq_0 \text{ZFC} + \text{IC} \leq_0 \text{ZFC} + \text{WC} \leq_0 \text{ZFC} + \text{MC} \leq_0 \text{ZFC} + \text{SC},$$

in a very strong sense, given that the properties are implied downwards.

We can also define

$$\Phi \equiv_0 \Psi : \Longleftrightarrow \Phi \leq_0 \Psi \wedge \Psi \leq_0 \Phi,$$

i.e. $C_\Phi = C_\Psi$, and

$$\Phi <_0 \Psi \Longleftrightarrow \Phi \leq_0 \Psi \wedge \Psi \not\equiv_0 \Phi.$$

Our results so far show

$$\text{ZFC} <_0 \text{ZFC} + \text{IC} <_0 \text{ZFC} + \text{WC}.$$

This hierarchy inherits complexity from the fact that it is just a subset relation. If Φ is incomplete, then there is some φ such that

$$\Phi \not\vdash \varphi \text{ and } \Phi \not\vdash \neg\varphi.$$

Thus $\Phi \cup \{\varphi\}$ and $\Phi \cup \{\neg\varphi\}$ are incomparable in $\leq_0$. In particular, we cannot compare $\text{ZFC} + \text{CH}$ and $\text{ZFC} + \neg\text{CH}$.

Answer (Second Attempt). Let Φ be a **cardinal property**, i.e. if $\Phi(x)$ then x is a cardinal, and write

$$\Phi C := \exists x. \Phi(x).$$

Try to formalise the idea of "Φ-cardinals are bigger than Ψ-cardinals". We clearly can't possibly say "every Φ-cardinal is bigger than every Ψ-cardinal". Let ι_Φ be the smallest Φ-cardinal, if it exists, and define

$$\Phi \leq_1 \Psi : \Longleftrightarrow (\Phi C + \Psi C \implies \iota_\Phi < \iota_\Psi),$$

and similarly for $<_1, \equiv_1$. Theorem 5.19 shows that that $\text{IC} <_1 \text{WC}$.

Lecture 10

Question. How can we make ZFC into a large cardinal axiom?

Answer (ES#1.3). Use AI:

$$\begin{aligned} \text{inf}(\kappa) &:= \kappa \text{ is an infinite cardinal} \\ \text{ZFC} &\iff \text{ZFC} + \text{inf } C, \end{aligned}$$

and thus

$$\begin{aligned} \text{ZFC} + \text{inf } C &<_1 \text{ZFC} + \text{IC} \\ &<_1 \text{ZFC} + \text{WC} \\ &\leq_1 \text{ZFC} + \text{MC} \\ &\leq_1 \text{ZFC} + \text{SC}. \end{aligned}$$

Example 6.1 (Silly). We define silly cardinals

$$\Sigma(\kappa) := \iff \exists \lambda. \lambda \text{ is weakly compact and } \kappa \text{ is inaccessible,}$$

and hence

$$\text{ZFC} + \Sigma C \iff \text{ZFC} + \text{WC}.$$

In particular,

$$\begin{aligned} C_{\text{ZFC} + \Sigma C} &= C_{\text{ZFC} + \text{WC}} \\ \text{ZFC} + \Sigma C &\equiv_0 \text{ZFC} + \text{WC}. \end{aligned}$$

However, if $\Sigma C + \text{WC}$ holds, then

$$\begin{aligned} i_{\Sigma C} &< i_{\text{WC}} \\ \implies \text{ZFC} + \Sigma C &<_1 \text{ZFC} + \text{WC}. \end{aligned}$$

Example 6.2 (More Silliness). Define

$$\begin{aligned} \Sigma'(\kappa) : \iff &\neg \text{WC} \implies \exists \alpha (\aleph_\alpha \text{ inaccessible} \wedge \kappa = \aleph_{\alpha+\omega}) \\ &\wedge \text{WC} \implies \exists \alpha (\aleph_\alpha \text{ weakly compact} \wedge \kappa = \aleph_{\alpha+\omega}). \end{aligned}$$

Clearly

$$\begin{aligned} \text{ZFC} + \Sigma' C &\iff \text{ZFC} + \text{IC} \\ \implies C_{\text{ZFC} + \Sigma' C} &= C_{\text{ZFC} + \text{IC}} \\ \implies \text{ZFC} + \Sigma' C &\equiv_0 \text{ZFC} + \text{IC}, \end{aligned}$$

and thus if $\Sigma C + \text{WC}$ holds, then

$$\begin{aligned} i_{\text{WC}} &< i_{\Sigma' C} \\ \implies \text{ZFC} + \text{WCC} &<_1 \text{ZFC} + \Sigma'. \end{aligned}$$

Aim. Neither $\leq_0$ nor $\leq_1$ work sensibly. We need some notion like $<_0$ but with less incomparability.

Remark. This phenomenon is called an **identity crisis**: a situation where we have two large cardinal axioms ΦC and ΨC where ΦC is clearly stronger than ΨC , but

$$\Phi C + \Psi C \wedge i_\Phi = i_\Psi,$$

is consistent [Mag76, HMP20].

6.2 Consistency Strength Hierarchy

Assumption. For sake of simplicity, we focus on theories of the type

$$\text{ZFC} + A,$$

where A is a single axiom.

Definition 6.3 (Consistency Statements). We define the **consistency statements**

$$\text{Cons} := \{\text{Cons}(\text{ZFC} + A); A \text{ is a formula}\},$$

where the formulae $\text{Cons}(\text{ZFC} + A)$ are all arithmetical formulas, modulo coding.

Remark 6.4. Since all quantifiers are bounded by $\mathbb{N}$, consistency statements are Δ_0 formulae and thus they're absolute between transitive models of set theory.

Definition 6.5 (Consistency Strength Hierarchy). Define

$$A \leq_{\text{Cons}} B : \iff \text{Cons} \cap C_{\text{ZFC}+A} \subseteq \text{Cons} \cap C_{\text{ZFC}+B}.$$

Question. Does this remove the problems?

Answer. We have the following advantages:

- (i) Silly cardinals problems go away, since for Σ, Σ' ,

$$\begin{aligned} C_{\text{ZFC}+\Sigma C} &= C_{\text{ZFC}+\text{WC}} \\ \implies \text{Cons} \cap C_{\text{ZFC}+\Sigma C} &= \text{Cons} \cap C_{\text{ZFC}+\text{WC}} \\ \implies \text{WC} &\equiv_{\text{Cons}} \Sigma C, \end{aligned}$$

and similarly, $\text{IC} \equiv_{\text{Cons}} \Sigma' C$.

- (ii) Note that when we proved Theorem 5.8 ($\text{IC} <_0 \text{WC}$), we showed

$$\text{ZFC} + \text{WC} \vdash \text{Cons}(\text{ZFC} + \text{IC}),$$

and so we still retain

$$\text{ZFC} <_{\text{Cons}} \text{ZFC} + \text{IC} <_{\text{Cons}} \text{ZFC} + \text{WC}.$$

- (iii) For non-linearity, both Gödel's proof of consistency of CH [Gö38], and Cohen's method of forcing [Coh63], proof of consistency of $\neg\text{CH}$, produce **transitive** models of ZFC. Therefore any consistency statements true before application of the method will be true after the application of the method. This implies that

$$\text{Cons} \cap C_{\text{ZFC}} = \text{Cons} \cap C_{\text{ZFC}+\text{CH}} = \text{Cons} \cap C_{\text{ZFC}+\neg\text{CH}},$$

and thus

$$\text{ZFC} \equiv_{\text{Cons}} \text{ZFC} + \text{CH} \equiv_{\text{Cons}} \text{ZFC} + \neg\text{CH}.$$

Remark. We're not completely free from identity crises. Non-linearity and ill-foundedness can be shown using Rosser's sentences [Ham21].

7 Measurability, Elementarity and Reflection

Assumption. For the sake of simplicity, we first work under the assumption that there are $\kappa < \lambda$ such that κ is measurable and λ is inaccessible. In particular, $V_\lambda \models \text{ZFC} + \text{MC}$, so our assumptions are strongly stronger than $\text{ZFC} + \text{MC}$, in the sense of $\leq_{\text{Cons}}$.

Remark. We will remove the additional inaccessible later.

Fix U a κ -complete non-trivial ultra filter on κ . Using our ultrapowers Section 5.1, we can form $\text{Ult}(V_\lambda, U)$.

Remark. Recall that $\text{Ult}(V_\lambda, U)$ consists of equivalence classes of choice functions $f : \kappa \rightarrow V_\lambda$ where

$$f \sim_U g : \iff \{\alpha \in \kappa; f(\alpha) = g(\alpha)\} \in U.$$

If f is such a function, by Regularity of κ , $\text{ran}(f)$ is bounded in V_λ and therefore $f \in V_\lambda$, though this does not necessarily hold for the entire class. Thus if we pick for each $\sim_U$ -equivalence class a representative and call this set X , then $X \subseteq V_\lambda$.

Question. What is the $\in$ -relation in X ?

Answer. It is induced by the ultrapower as

$$f E g : \iff \{\alpha \in \kappa; f(\alpha) \in g(\alpha)\} \in U.$$

Clearly, $E \neq \in$.

Remark. To summarise, we found $X \subseteq V_\lambda$ and $E \subseteq X \times X$, such that the map

$$x \longmapsto c_x,$$

where c_x is the constant function with value x , is an elementary embedding, from

$(V_\lambda, \in)$ to (X, E) . We would like to have some transitive set $M \subseteq V_\lambda$ such that

$$(X, E) \cong (M, \in),$$

through some isomorphism π , and then

$$\begin{array}{ccccc} V_\lambda & \longrightarrow & X & \longrightarrow & M \\ x & \longmapsto & c_x & \longmapsto & \pi(c_x) \end{array}$$

To use Mostowski 5.17, we need extensional and well-founded. Extensionality is free, since $V_\lambda \models \text{Extensionality}$ and thus so is (X, E) . Well-foundedness requires a proof, using measurability of κ .

Lecture 11

Theorem 7.1. (X, E) is well-founded.

Proof. Suppose not, then there is a sequence f_n of functions decreasing in E such that

$$[f_{n+1}]E[f_n] \iff A_n := \{\alpha; f_{n+1}(\alpha) \in f_n(\alpha)\} \in U.$$

So, by κ -completeness, we have

$$A := \bigcap_{n \in \mathbb{N}} A_n = \{\alpha; \forall n \in \mathbb{N}. f_{n+1}(\alpha) \in f_n(\alpha)\} \in U.$$

Since $A \in U$, $A \neq \emptyset$, so choose $\alpha \in A$ and get for all $n \in \mathbb{N}$, $f_{n+1}(\alpha) \in f_n(\alpha)$. But this contradicts well-foundedness of V_λ . $\square$

Remark. Note that

- (i) We only needed $\aleph_1$ -completeness.
- (ii) We proved non-existence of decreasing sequences, which is only equivalent to well-foundedness assuming AC.
- (iii) $\aleph_1$ -completeness is equivalent to non-existence of decreasing sequences (ES#3).

Idea. Using Mostowski's 5.17, we can collapse X to a transitive set M such that

$$\pi : (X, E) \cong (M, \in).$$

Notation. If $f : \kappa \rightarrow V_\lambda$, we write

$$(f)_U := \pi([f]_U),$$

omitting U if clear. Concatenating the two embeddings, we get

$$\begin{array}{l} j := j_U : V_\lambda \longrightarrow M \\ x \longmapsto (c_x)_U. \end{array}$$

Aim. We want to understand this embedding.

Proposition 7.2. $|M| = \lambda$.

Proof. We have

$$\{[c_\alpha]; \alpha < \lambda\} \subseteq X \subseteq V_\lambda,$$

thus

$$|\{[c_\alpha]; \alpha < \lambda\}| \leq |X| \leq |V_\lambda| = \lambda,$$

since λ is inaccessible. So $|X| = \lambda$, and thus $|M| = \lambda$, since $(X, E) \cong (M, \in)$. $\square$

Proposition 7.3. If $x \in M$, then $|x| < \lambda$.

Proof. Suppose $x \in M$, so we have some $f : \kappa \rightarrow V_\lambda$ such that $x = (f)$. By Regularity of λ , we find α such that $f : \kappa \rightarrow V_\alpha$. If $y \in x$, by transitivity of M , $y \in M$, so there is $g : \kappa \rightarrow V_\lambda$ such that $(g) = y$. Since $y \in x$, we have $(g) \in (f)$, so

$$[g]E[f] \iff \{\alpha; g(\alpha) \in f(\alpha)\} \in U.$$

Hence WLOG, we can assume $g : \kappa \rightarrow V_\alpha$. Therefore

$$|x| \leq |V_\alpha|^\kappa.$$

Since λ is inaccessible, $|V_\alpha| < \lambda$ by Hausdorff's Theorem 2.11, thus $|x| < \lambda$. $\square$

Proposition 7.4 (Figure 3). $M \subseteq V_\lambda$.

Proof. By Position 7.3, all elements of M are λ -small. Since M is transitive, this means, $|\text{tcl}(x)| < \lambda$ if $x \in M$, so $M \subseteq H_\lambda = V_\lambda$ by ES#1.7. $\square$

Proposition 7.5. $\text{Ord} \cap M = \lambda$.

Proof. Remember $\{[c_\alpha]; \alpha < \lambda\} \subseteq X$, and thus $\{(c_\alpha); \alpha < \lambda\} \subseteq M$. But this is a set of ordinals of order type λ . Thus $\lambda \leq \text{Ord} \cap M$. But by Proposition 7.4, $\text{Ord} \cap M \leq \lambda$. $\square$

Remark 7.6. If $f : \kappa \rightarrow \lambda$, then (f) is an ordinal, since

$$\kappa = \{\alpha; f(\alpha) \text{ is an ordinal}\} \in U,$$

since by Los' 5.10:

$$(X, E) \models [f] \text{ is an ordinal}$$

$$(M, \in) \models (f) \text{ is an ordinal}.$$

Being an ordinal is absolute for transitive models of ZFC.

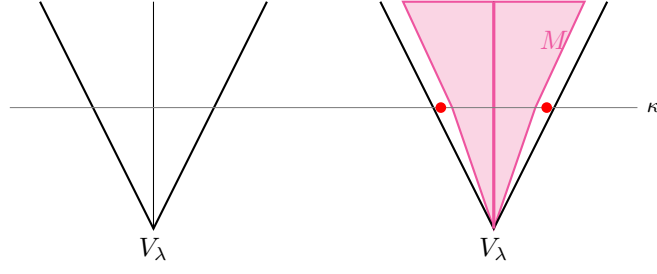


Figure 3: V_λ as per Proposition 7.5. We will see that the red corners, where M might miss part of V_κ , cannot exist.

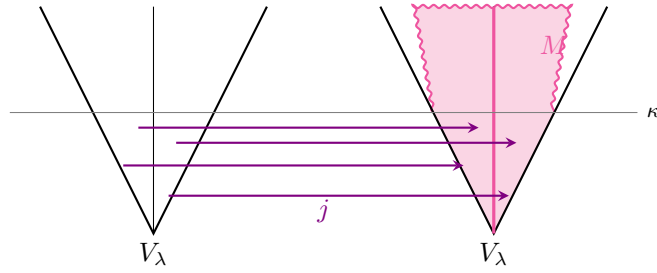


Figure 4: Improvement on Figure 3 given by Proposition 7.7.

Question. Do the corners in Figure 3 really exist?

Proposition 7.7 (Figure 4). $j \upharpoonright V_\kappa = \text{id}$.

Proof. We prove this by $\in$ -induction on V_κ . Suppose $x \in V_\kappa$ arbitrary such that the claim is true for all $y \in x$. Then

$$\begin{aligned} y \in x &\iff (M, \in) \models j(y) \in j(x) \\ &\iff (M, \in) \models y \in j(x) \quad (\text{I.H.}) \\ &\iff y \in j(x). \end{aligned}$$

If $y \in x$, then $y \in j(x)$. Thus $x \subseteq j(x)$. If now $(f : \kappa \rightarrow V_\lambda) \in j(x) = (c_x)$, then

$$\bigcup_{y \in x} \{\alpha; f(\alpha) = y\} = \{\alpha; f(\alpha) \in x\} = \{\alpha; f(\alpha) \in c_x(\alpha)\} \in U,$$

as the union is of size $|x| < \kappa$ and κ is measurable, thus it is inaccessible. Hence

$$\begin{aligned} \exists y \in x : \{\alpha; f(\alpha) = y\} &\in U \\ \implies (f) = (c_y) = (j(y)) &= (y), \end{aligned}$$

by induction hypothesis. This gives $j(x) \subseteq x$. Together, $j(x) = x$. □

Proposition 7.8 (Figure 5). $j(\kappa) > \kappa$.

Proof. As seen before, $j(\kappa)$ is an ordinal. Also clearly $\kappa \leq j(\kappa)$. We prove the claim by providing an ordinal that sits between all $\alpha < \kappa$ and $j(\kappa)$. Consider $(\text{id} : \kappa \rightarrow V_\lambda)$, which is an ordinal by Remark 7.6. Fix $\gamma < \kappa$ and compare (id) with γ :

$$\begin{aligned} A_\gamma &:= \{\alpha; \text{id}(\alpha) > c_\gamma(\alpha)\} \\ &= \{\alpha; \alpha > \gamma\} = \kappa \setminus \gamma \in U. \end{aligned}$$

Therefore

$$(\text{id}) > (c_\gamma) = j(\gamma) = \gamma,$$

by Proposition 7.7. This shows $(\text{id}) \geq \kappa$, since $\gamma < \kappa$ was arbitrary. Also consider

$$\{\alpha; \text{id}(\alpha) < c_\kappa(\alpha)\} = \{\alpha \in \kappa; \alpha < \kappa\} = \kappa \in U.$$

Hence $(\text{id}) < (c_\kappa) = j(\kappa)$. Putting together, we get $\kappa \leq (\text{id}) < j(\kappa)$. $\square$

Definition 7.9 (Non-Trivial Embedding). An embedding $j : V_\lambda \rightarrow M$ is called **non-trivial** if $j \neq \text{id}$.

Proposition 7.10 (ES#3). For every non-trivial embedding there is an ordinal γ such that $j(\gamma) > \gamma$.

Definition 7.11 (Critical Point). Let $j : V_\lambda \rightarrow M$ be a non-trivial embedding. We call the **critical point** of j the least ordinal γ such that $j(\gamma) > \gamma$ and denote it $\text{crit}(j)$.

Lecture 12

Summary of our situation as depicted in Figure 5:

(i) We have an elementary embedding

$$\begin{aligned} j &= j_U : V_\lambda \longrightarrow M \\ x &\longmapsto (c_x), \end{aligned}$$

(ii) $M \subseteq V_\lambda$ with $\text{Ord} \cap M = \lambda$,

(iii) $j \upharpoonright V_\kappa = \text{id} \upharpoonright V_\kappa$,

(iv) $\kappa \leq (\text{id}) < j(\kappa)$,

(v) κ is the critical point of j .

Theorem 7.12 (Fundamental for Measurable Cardinals). Let λ be inaccessible and $\kappa < \lambda$. Then the following are equivalent

(i) κ is measurable.

(ii) There is a transitive $M \subseteq V_\lambda$ and a non-trivial embedding $j : V_\lambda \rightarrow M$ with critical point κ .

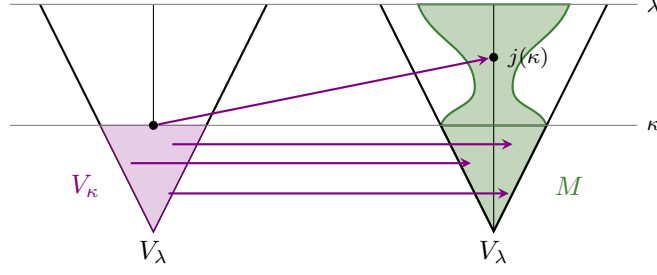


Figure 5: State of the art over Figure 4 given by Proposition 7.8.

Proof. The direction (i) $\implies$ (ii) is given by Proposition 7.8. We now do the direction (ii) $\implies$ (i). First, we observe that if j is an embedding from V_λ to M with critical point κ , i.e. $\text{crit}(j) = \kappa$, then $j \upharpoonright V_\kappa = \text{id} \upharpoonright V_\kappa$, by an easy induction on rank.

Define $U \subseteq \mathcal{P}(\kappa)$ by

$$X \in U : \iff X \subseteq \kappa \wedge k \in j(X).$$

We wish to show that U is a κ -complete non-trivial ultrafilter on κ . We have

(i) Non-triviality:

$$j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\},$$

by elementarity and $k = \text{crit}(j)$. So for any $\alpha < \kappa$, $k \notin j(\{\alpha\})$.

(ii) Filter: we have, by elementarity,

$$\begin{aligned} j(X \cap Y) &= j(X) \cap j(Y), \\ X \subseteq Y &\implies j(X) \subseteq j(Y), \\ j(\emptyset) &= \emptyset. \end{aligned}$$

So we get directly that

$$\begin{aligned} X, Y \in U &\implies X \cap Y \in U \\ X \subseteq Y, X \in U &\implies Y \in U \\ \emptyset &\notin U. \end{aligned}$$

Because $\kappa = \text{crit}(j)$, we have $\kappa < j(\kappa)$, hence $\kappa \in j(\kappa)$, and thus $\kappa \in U$.

(iii) Ultrafilter: by elementarity, we have

$$j(\kappa \setminus X) = j(\kappa) \setminus j(X).$$

So if $\kappa \notin j(X)$, then $\kappa \in j(\kappa \setminus X)$.

(iv) κ -Completeness: fix some $\delta < \kappa$, a sequence $\mathcal{A} := (A_\alpha \in U; \alpha < \delta)$ and $k \in j(A_\alpha)$. We want to show $\bigcap_{\alpha < \delta} A_\alpha \in U$. But

$$\begin{aligned} V_\lambda \models \mathcal{A} \text{ is a } \delta\text{-sequence of subsets of } \kappa \\ \implies \mathcal{M} \models j(\mathcal{A}) \text{ is a } j(\delta) = \delta\text{-sequence of subsets of } j(\kappa), \end{aligned}$$

by elementarity. Similarly,

$$\begin{aligned} V_\lambda \models A_\alpha \text{ is the } \alpha\text{-th element of } \mathcal{A} \\ \implies \mathcal{M} \models j(A_\alpha) \text{ is the } j(\alpha) = \alpha\text{-th element of } j(\mathcal{A}). \end{aligned}$$

To summarise we have $j(\mathcal{A}) = (j(A_\alpha); \alpha < \delta)$, and thus

$$A := \bigcap_{\alpha < \delta} A_\alpha \implies j(A) = \bigcap_{\alpha < \delta} j(A_\alpha),$$

by what we just showed. Therefore $\kappa \in j(A)$, and thus $A \in U$.

□

Idea. Proof breakdown:

- (i) j restricts to id on V_κ ,
- (ii) Define filter by $X \subseteq \kappa \wedge \kappa \in j(X)$,
- (iii) Non-triviality and filter by elementarity and $\kappa = \text{crit}(j)$,
- (iv) Ultra by elementarity on complements,
- (v) κ -Completeness by elementarity on intersecting δ -sequences for $\delta < \kappa$.

Remark. There is a remarkable similarity of this proof and the one of Theorem 5.7.

7.1 Observations about Cardinals in $\mathcal{M}$

Remark. Let us remark that elementarity buys us the following consequence

$$\begin{aligned} V_\lambda \models \kappa \text{ is measurable,} \\ \implies \mathcal{M} \models j(\kappa) \text{ is measurable.} \end{aligned}$$

In particular, if $k < \alpha < j(\kappa)$, then there is some $\alpha < \mu < j(k)$ such that

$$\mathcal{M} \models \mu \text{ is inaccessible.}$$

Remark. Since these are all between κ and λ . this is a first hint that we may be in the situation of Figure 5, with $\mathcal{M} \neq V_\lambda$. Indeed, if λ happens to be the least inaccessible above κ , then for every $\kappa < \mu < \lambda$, we have

$$V_\lambda \models \mu \text{ is not inaccessible.}$$

Since $\mathcal{M} \models \mu$ is inaccessible, for some of these, we have, in this case, $\mathcal{M} \neq V_\lambda$.

Remark. Since M is transitive in V_λ , we know that the notions *cardinal*, *regular*, *inaccessible* are downwards absolute. So if

$$\begin{aligned} V_\lambda \models \mu \text{ is a cardinal/regular/inaccessible,} \\ \implies \mathcal{M} \models \mu \text{ is a cardinal/regular/inaccessible.} \end{aligned}$$

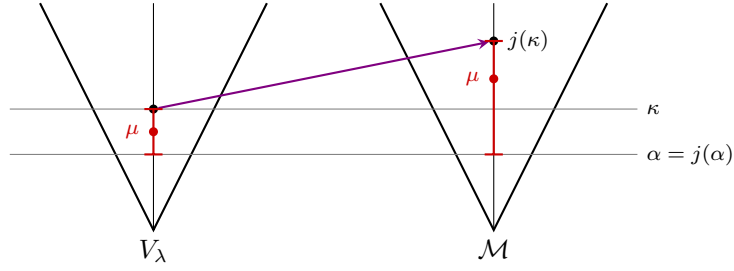


Figure 6: Reflection argument for embeddings into ultrapower collapse.

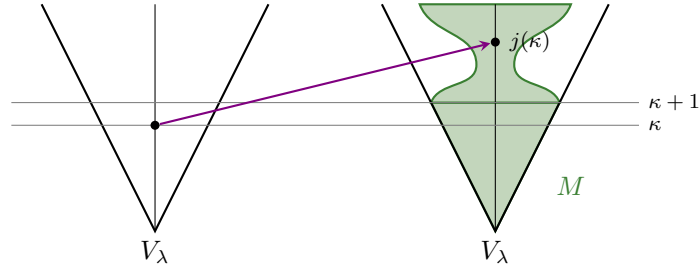


Figure 7: Update on Figure 5.

This is certainly true for the inaccessible cardinal κ . Thus

$$\mathcal{M} \models \kappa \text{ is inaccessible.}$$

This allows us to do a reflection argument, with situation as in Figure 6:

$$\begin{aligned} \mathcal{M} &\models \exists \mu : j(\alpha) < \mu < j(\kappa) \wedge \mu \text{ inaccessible,} \\ \implies V_\lambda &\models \exists \mu : \alpha < \mu < \kappa \wedge \mu \text{ inaccessible,} \end{aligned}$$

by elementarity. This yields a new proof of Theorem 4.13.

Theorem 7.13 (Figure 7). $V_{\kappa+1} \subseteq M$.

Proof. Note that we cannot expect $j \upharpoonright V_{\kappa+1} = \text{id} \upharpoonright V_{\kappa+1}$, since $\kappa \in V_{\kappa+1}$, but $j(\kappa) \neq \kappa$. Let us also note that if $X \in V_{\kappa+1}$, then $X \subseteq V_\kappa$.

Fix one such $X \in V_{\kappa+1}$. It suffices to show $X = j(X) \cap V_\kappa$, as it implies that $X \in M$.

To show $\subseteq$, if $x \in X$, then

$$\begin{aligned} x \in V_\kappa &\implies j(x) = x \\ &\implies j(x) \in j(X) \quad (\text{by elem.}) \\ &\implies x \in j(X) \cap V_\kappa. \end{aligned}$$

To show $\supseteq$, if $x \in j(X) \cap V_\kappa$, then

$$\begin{aligned} x \in j(X) \wedge x \in V_\kappa &\implies j(x) = x \\ &\implies j(x) \in j(X) \\ &\implies x \in X \quad (\text{by elem.}). \end{aligned}$$

□

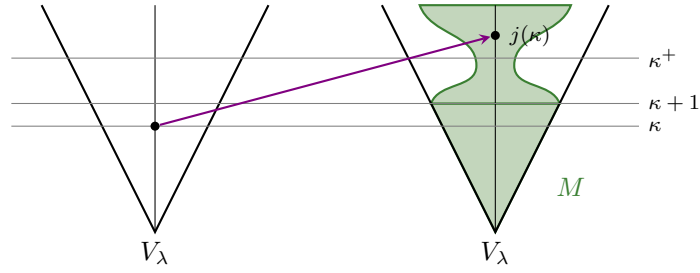


Figure 8: Update on Figure 7.

Corollary 7.14 (Figure 8). If κ^+ is the successor of κ in V_λ (in "reality") then

$$\mathcal{M} \models \kappa^+ \text{ is the successor of } \kappa,$$

e.g. if $\kappa < \alpha < \kappa^+$, then α is not a cardinal in $\mathcal{M}$.

Proof. Suppose $\kappa < \alpha < \kappa^+$. Then there is a well-order of order type α on κ . This well-order is a subset of V_κ , hence it is an element of $V_{\kappa+1}$, and thus by Theorem 7.13, an element of $\mathcal{M}$. Hence

$$\mathcal{M} \models \alpha \text{ is not a cardinal.}$$

□

Question. How big is $j(\kappa)$ really?

Proposition 7.15. $|j(\kappa)| \leq 2^\kappa$.

Proof. We just count the elements of $j(\kappa)$: if (f) is an ordinal such that

$$(f) \in j(\kappa) = (c_\kappa),$$

then WLOG, $f : \kappa \rightarrow \kappa$. Thus there are only 2^κ many such functions. □

Corollary 7.16. $j(\kappa)$ is not a strong limit cardinal.

Corollary 7.17. $j(\kappa)$ is not inaccessible.

Corollary 7.18. $V_\lambda \neq M$.

Aim. Produce a concrete of a set in $V_\lambda \setminus M$. This will be the set U .

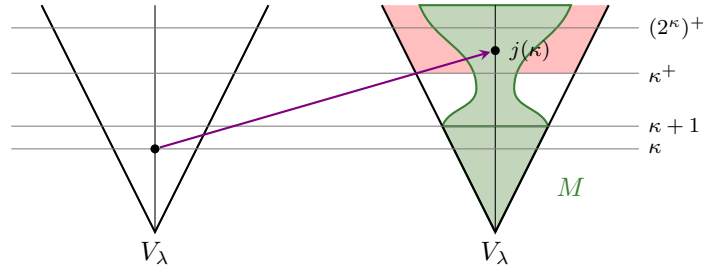


Figure 9: Update on Figure 8.

Theorem 7.19. $U \notin M$.

Proof. Suppose towards a contradiction that $U \in M$. First, we note that if $f : \kappa \rightarrow \kappa$, then $f \in V_{\kappa+1}$ and so $f \in M$. Therefore $\kappa^\kappa \in M$.

Remember that in V_λ ,

$$j(\kappa) = \{(f)_U; f : \kappa \rightarrow \kappa\}.$$

Thus $f \mapsto (f)_U$ is a surjection from κ^κ onto $j(\kappa)$. Therefore, suffices to show that this function exists in M .

Since we assumed $U \in M$, the equivalence relation $\sim_U$ on κ^κ defined by

$$f \sim_U g : \iff \{\alpha; f(\alpha) = g(\alpha)\} \in U,$$

is an element of M by Separation. Thus $[f]_U$ is an element of M , again by Separation. But the Mostowski Collapse is uniquely defined in models of ZFC, so in M we can define $f \mapsto (f)_U$, by $(f)_U$ is the unique image of the Mostowski Collapse of $[f]_U$. We apply Separation again to get that $f \mapsto (f)_U$ is an element of M , so

$$|j(\kappa)| \leq 2^\kappa,$$

in contradiction to

$$\mathcal{M} \models j(\kappa) \text{ is measurable.}$$

□

Idea. The proof shows failure of iterating the ultrapower on the same U . We use:

$$V_\lambda \models |j(\kappa)| \leq 2^\kappa,$$

and show that if U was in M , then the same would be true in $\mathcal{M}$. This contradicts

$$\mathcal{M} \models j(\kappa) \text{ is measurable.}$$

Corollary 7.20 (Figure 10). $V_{\kappa+2} \not\subseteq M$.

Proof. $U \in V_{\kappa+2}$ but by Theorem 7.19 $U \notin M$.

□

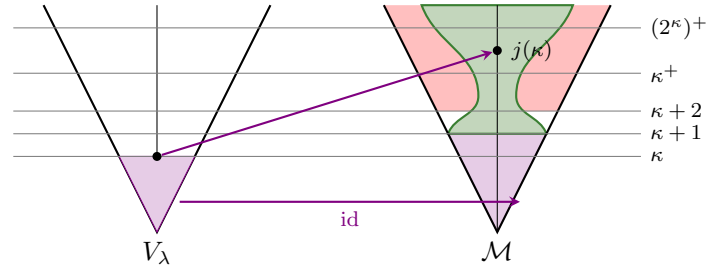
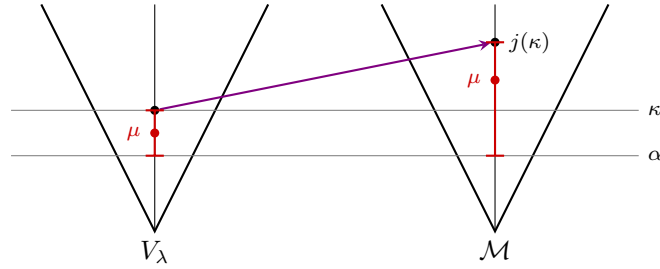

 Figure 10: The current state of affairs for $\mathcal{M}$.


Figure 11: Our familiar picture.

Remark. This clearly implies that κ cannot be measurable in $\mathcal{M}$ via the ultrafilter U . But it does not exclude the possibility that there is another ultrafilter U' on κ such that U' survives in the ultrapower, which would make κ measurable in $\mathcal{M}$.

Definition 7.21 (Surviving). A cardinal κ is called **surviving** if it is measurable with ultrapower embedding

$$j : V_\lambda \rightarrow M,$$

and $\mathcal{M} \models \kappa$ is measurable.

A reflection argument shows this is stronger than being measurable:

Corollary 7.22. If κ is surviving, then κ cannot be the least measurable cardinal.

Proof. Suppose κ is surviving. Then

$$\begin{aligned} \mathcal{M} &\models \exists \lambda < j(\kappa). \lambda \text{ is measurable} \\ \implies V_\lambda &\models \exists \lambda < \kappa. \lambda \text{ is measurable.} \end{aligned}$$

□

We can in fact prove something stronger:

Corollary 7.23 (Figure 11). If κ is surviving, then κ is the κ -th measurable cardinal.

Proof. Fix $\alpha < \kappa$, and argue by reflection that

$$\begin{aligned} & \mathcal{M} \models \exists \beta < j(\kappa). (\alpha < \beta \wedge \beta \text{ is measurable}) \\ \iff & \mathcal{M} \models \exists \beta < j(\kappa). (j(\alpha) < \beta \wedge \beta \text{ is measurable}) \\ \iff & V_\lambda \models \exists \beta < \kappa. (\alpha < \beta \wedge \beta \text{ is measurable}). \end{aligned}$$

Thus the set of measurable cardinals below κ is unbounded. $\square$

7.2 Weak Compactness in Ultrapower

Question. Is κ weakly compact in M ?

Idea. If "YES", we have resolved our question about the relative strength measurability versus weak compactness, by the reflection argument:

Corollary 7.24. If κ is measurable and $\mathcal{M} \models \kappa$ is weakly compact, then κ is the κ -th weakly compact cardinal.

Proof. Fix $\alpha < \kappa$, then

$$\begin{aligned} & \mathcal{M} \models \kappa \text{ is weakly compact} \\ \implies & \mathcal{M} \models \exists \beta < j(\kappa). (j(\alpha) < \beta \wedge \beta \text{ is weakly compact}) \\ \implies & V_\lambda \models \exists \beta < \kappa. (\alpha < \beta \wedge \beta \text{ is weakly compact}) \end{aligned}$$

Hence the set of weakly compact cardinals below κ is unbounded. $\square$

Corollary 7.25. If α is any ordinal below the least measurable, then the α -th weakly compact is not measurable.

Theorem 7.26. $\mathcal{M} \models \kappa$ is weakly compact.

Proof. Let L be an $\mathcal{L}_{\kappa, \kappa}$ -language with set of non-logical symbols $S = \{s_\alpha; \alpha < \kappa\} \subseteq V_\kappa$ and Φ a set of L -formulas such that

$$\mathcal{M} \models \Phi \text{ is } \kappa\text{-satisfiable.}$$

Since κ is inaccessible, by Lemma 5.14 we have $|L| \leq \kappa$ and thus we may take WLOG

$$\Phi = \{\varphi_\alpha; \alpha < \kappa\} \subseteq V_\kappa.$$

For each $\Phi_0 \subseteq \Phi$ such that $|\Phi_0| < \kappa$, there is a model $N_{\Phi_0} \in M$ such that $\mathcal{N}_{\Phi_0} \models \Phi_0$. Since $M \subseteq V_\lambda$, $\mathcal{N}_{\Phi_0} \in V_\lambda$ and since " $\mathcal{N}_{\Phi_0} \models \Phi_0$ " is a bounded formula.

$$V_\lambda \models \Phi \text{ is } \kappa\text{-satisfiable.}$$

By Theorem 5.15, since κ is measurable, κ is weakly-compact in V_λ and so

$$V_\lambda \models \Phi \text{ is satisfiable,}$$

and thus there exists a model $N \in V_\lambda$ such that $\mathcal{N} \models \Phi$.

We want to show that we can take $N \in M$. If so, then again by absoluteness of $\mathcal{N} \models \Phi$, we have that $\mathcal{M} \models \Phi$ is satisfiable. By elementarity

$$\begin{aligned} & V_\lambda \models N \text{ is an } L\text{-structure and } \forall \varphi \in \Phi. " \mathcal{N} \models \varphi " \\ \implies & \mathcal{M} \models j(N) \text{ is a } j(L)\text{-structure and } \forall \varphi \in j(\Phi). " j(\mathcal{N}) \models \varphi ". \end{aligned}$$

We note that $j(L)$ is an $\mathcal{L}_{j(\kappa), j(\kappa)}$ -language with non-logical symbols $j(S)$. However,

$$\begin{aligned} & V_\lambda \models s_\alpha \text{ is the } \alpha\text{-th symbol in } S \\ \implies & \mathcal{M} \models j(s_\alpha) \text{ is the } j(\alpha)\text{-th symbol in } j(S) \\ \implies & \mathcal{M} \models s_\alpha \text{ is the } \alpha\text{-th symbol in } j(S), \end{aligned}$$

since $s_\alpha \in V_\kappa$ and $j \upharpoonright V_\kappa = \text{id}$. Therefore, $S \subseteq j(S)$ and we have an S -reduct of the $j(S)$ -structure $j(\mathcal{N})$, which we denote $\overline{\mathcal{N}}$.

Similarly, we understand $j(\Phi)$ by looking at

$$\begin{aligned} & V_\lambda \models \alpha\text{-th formula in } \Phi \text{ is } \varphi_\alpha \\ \implies & \mathcal{M} \models j(\alpha)\text{-th formula in } j(\Phi) \text{ is } j(\varphi_\alpha) \\ \implies & \mathcal{M} \models \alpha\text{-th formula in } j(\Phi) \text{ is } \varphi_\alpha, \end{aligned}$$

which gives $\Phi \subseteq j(\Phi)$. Putting everything together:

$$\begin{aligned} & \mathcal{M} \models "j(N) \models j(\Phi)" \\ \implies & \mathcal{M} \models "\overline{\mathcal{N}} \models j(\Phi)" \\ \implies & \mathcal{M} \models "\overline{\mathcal{N}} \models \Phi", \end{aligned}$$

so $\mathcal{M} \models \Phi$ is satisfiable. □

Remark. An alternative route is given by putting a size bound on N to show $N \in M$, in the spirit of a Löwenheim-Skolem Theorem. The argument goes as follows: suppose $|N| \leq \kappa$, then we can think of N as a structure on κ

$$\mathcal{N} = (\kappa, -),$$

where $-$ are the interpretations of L -symbols.

- (i) The relations would be subsets of κ^n ,
- (ii) The functions would be functions $\kappa^n \rightarrow \kappa$,
- (iii) Constants: elements of κ .

This would be something that lives in $V_{\kappa+1}$ and therefore in M . So, there is an isomorphic copy of $\mathcal{N}$ that lives in the part of V_λ that is preserved in $\mathcal{M}$ and so $\mathcal{M}$ contains that isomorphic copy, thus

$$\mathcal{M} \models \Phi \text{ is satisfiable.}$$

8 Strengthenings of Measurability

Idea. Four possible directions:

- (i) Limit processes,
- (ii) Survival,
- (iii) Strength,
- (iv) Supercompactness.

8.1 Limit Processes

Aim. We wish to show this doesn't generate any new ideas.

Definition 8.1. We say κ is

- (i) **0-measurable** if it is measurable,
- (ii) **$(\alpha + 1)$ -measurable** if it is α -measurable and there are unboundedly many α -measurables below κ ,
- (iii) κ is **λ -measurable**, with λ -limit, if for all $\alpha < \kappa$, κ is α -measurable.

Theorem 8.2 (Surviving κ -Measurable). If κ is surviving, then κ is κ -measurable.

Proof. This is exactly Corollary 7.23 applied inductively. □

8.2 Survival

Idea. Survival is really a relation on ultrafilters on κ .

Definition 8.3 (Survival). We say V **survives** U if

$$U < V : \iff V \in M_U = \pi[\text{Ult}(V_\lambda, U)].$$

Remark. κ is surviving iff $\exists U, V$ ultrafilters on κ with $U < V$.

Question. What kind of statement is $U < V$?

Answer. It means that there is $g : \kappa \rightarrow V_\lambda$ such that $(g)_U = V$. Let $f : \kappa \rightarrow \kappa$ be such that $(f)_U = \kappa$. Then by Los' Theorem 5.10

$$\begin{aligned} \mathcal{M}_U \models (g)_U \text{ is a } (f)_U\text{-complete u.f. on } (f)_U \\ \iff \{\alpha; g(\alpha) \text{ is an } f(\alpha)\text{-complete u.f. on } f(\alpha)\} \in U. \end{aligned}$$

Note that $f(\alpha) < \kappa$, so $g(\alpha) \in V_\kappa$.

Let us extract some consequences:

Remark 8.4. If $\mathcal{M}$ is any inner model such that $V_{\kappa+1} \subseteq M$ and $U, V \in M$, then

$$\mathcal{M} \models U < V \iff V_\lambda \models U < V.$$

Definition 8.5 (Mitchell Rank). A cardinal κ has **Mitchell rank** above n , write $o(\kappa) \geq n$ if there are k -complete non-trivial ultrafilters $U_0, U_1, \dots, U_n$ such that

$$U_0 < U_1 < \dots < U_n.$$

Moreover, we say that κ has Mitchell rank n if $o(\kappa) \geq n$ but not $o(\kappa) \geq n+1$.

Remark. This definition can also be, *carefully*, extended into the transfinite.

Remark. Clearly $o(\kappa) \geq 0$ iff κ is measurable, and $o(\kappa) \geq 1$ iff κ is surviving.

Proposition 8.6. $o(\kappa) \geq n+1$ is strictly stronger than $o(\kappa) \geq n$.

Proof. We only show the result for $n=1$, as the rest follows by induction. If $o(\kappa) \geq 2$ we have $U_0 < U_1 < U_2$. Let M_0 be the ultrapower by U_0 , then $V_{\kappa+1} \subseteq M_0$, so by Remark 8.4, we get

$$\begin{aligned} \mathcal{M}_0 &\models U_1 < U_2 \\ \implies \mathcal{M}_0 &\models o(\kappa) \geq 1. \end{aligned}$$

We now apply a standard reflection argument

$$\begin{aligned} \mathcal{M}_0 &\models \exists \mu < j(\kappa). o(\mu) \geq 1 \\ \implies V_\lambda &\models \exists \mu < \kappa. o(\mu) \geq 1. \end{aligned}$$

Therefore, if $o(\kappa) \geq 2$, then κ is not the least cardinal with $o(\kappa) \geq 1$. $\square$

8.3 Strength

Idea. Stop talking about ultrafilters and start talking directly about the embedding.

Definition 8.7 (α -Strong Embedding). Let

$$j : V_\lambda \longrightarrow M,$$

be any elementary embedding with $\text{crit}(j) = \kappa$. We say j is α -**strong** if $V_{\kappa+\alpha} \subseteq M$.

Corollary 8.8. Let $j_U : V_\lambda \longrightarrow M_U$ be the ultrapower embedding. Then j_U is 1-strong but not 2-strong.

Proof. By Theorem 7.13, $V_{\kappa+1} \subseteq M_U$, whereas by Corollary 7.20, $V_{\kappa+2} \not\subseteq M_U$. $\square$

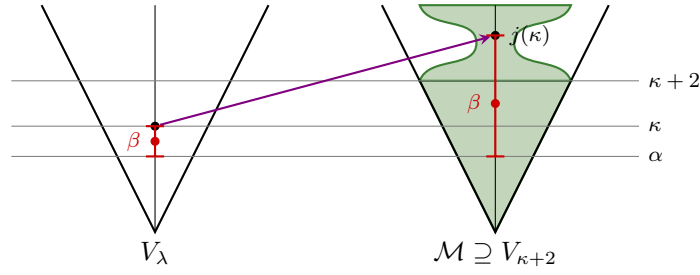


Figure 12: Reflection for 2-strong cardinals.

Definition 8.9 (α -Strong Cardinal). A cardinal κ is called **α -strong** if there is an α -strong embedding with critical point κ .

Corollary 8.10. If κ is measurable, κ is **1-strong** but not (*necessarily*) **2-strong**.

Proof. By Corollary 8.8, we have κ is 1-strong. For 2-strong, by Corollary 7.22, it suffices to take the least measurable cardinal. $\square$

Lecture 15

By reflection, we can show 2-strength to be *much* stronger than measurability:

Proposition 8.11 (Figure 12). If κ is 2-strong, then κ is not the least measurable.

Proof. Every 2-strong cardinal κ is measurable, by the reverse direction of the Fundamental Theorem 7.12. Thus there is a κ -complete non-trivial u.f. U on κ . But $U \in V_{\kappa+2} \subseteq M$. Hence fix $\alpha < \kappa$ and consider

$$\begin{aligned} & \mathcal{M} \models \kappa \text{ is measurable} \\ \implies & \mathcal{M} \models \exists \beta < j(\kappa). j(\alpha) < \beta \wedge \beta \text{ is measurable} \\ \implies & V_\lambda \models \exists \beta < \kappa. \alpha < \beta \wedge \beta \text{ is measurable.} \end{aligned}$$

$\square$

Theorem 8.12. If κ is 2-strong, then for each $n \in \mathbb{N}$, $o(\kappa) \geq n$.

Proof. Omitted, weaker result on ES#3.44 shows that if κ is 2-strong and $o(\kappa) \geq n$, then κ is not the least cardinal with $o(\kappa) \geq n$. $\square$

Definition 8.13 (Strong Cardinal). A cardinal κ is called **strong** if it is α -strong for all ordinals α .

Remark (Important). If κ is strong, then for each α , there is $j_\alpha : V_\lambda \rightarrow M_\alpha$ such that $V_{\kappa+\alpha} \subseteq M_\alpha$. It does not mean that this is witnessed by a single embedding!

Definition 8.14 (Reinhardt). A cardinal κ is called a **Reinhardt** if there is an embedding $j : V_\lambda \rightarrow M$ with $\text{crit}(j) = \kappa$ that is α -strong for all α , i.e. $M = V_\lambda$.

Kunen showed that there are no Reinhardt cardinals:

Theorem 8.15 (Kunen Inconsistency '71). Suppose $j : V \rightarrow M$ is elementary with $\text{crit}(j) = \kappa$. Define by recursion:

$$\begin{aligned} k_0 &:= \kappa, \\ k_{i+1} &:= j(k_i) \\ \hat{\kappa} &:= \bigcup_{i \in \mathbb{N}} k_i, \end{aligned}$$

i.e. $\hat{\kappa}$ is the least fixed point of j bigger than κ . Then there is $X \in V_{\hat{\kappa}+1}$ such that $X \notin M$, i.e. j is not $(\hat{\kappa} + 1)$ -strong.

Notation. If X is a totally ordered set, we write $[X]^\omega$ for the set of all strictly increasing functions from ω into X .

Definition 8.16 (ω -Jonsson). Let μ be any cardinal and $f : [\mu]^\omega \rightarrow \mu$. Then f is called ω -**Jonsson** if for all $X \subseteq \mu$ such that $|X| = \mu$, we have

$$\{f(y); y \in [X]^\omega\} = \mu.$$

Lemma 8.17 (Erdos-Hajnal '66). Every infinite cardinal has an ω -Jonsson function.

Proof. Omitted, see [Kan03b]. □

Proof of 8.15. Fix $j : V_\lambda \rightarrow M$. By Erdos-Hajnal Lemma 8.17, there exists an ω -Jonsson function $f : [\hat{\kappa}]^\omega \rightarrow \hat{\kappa}$. By elementarity,

$$\mathcal{M} \models j(f) \text{ is an } \omega\text{-Jonsson function on } j(\hat{\kappa}) = \hat{\kappa}.$$

Let $X := \{j(\alpha); \alpha \in \hat{\kappa}\} \subseteq V_{\hat{\kappa}}$, which can be done as $\hat{\kappa}$ is a j -fixed point. Clearly $|X| = \hat{\kappa}$.

Towards a contradiction, assume that $X \in M$. By definition, this means

$$\begin{aligned} \hat{\kappa} &= \{j(f)(y); y \in [X]^\omega\} \\ &= \{j(f(x)); x \in [\hat{\kappa}]^\omega\} \\ &= \{j(\alpha); \alpha \in \hat{\kappa}\}, \end{aligned}$$

by the fact that f is ω -Jonsson.

Now if $y \in [X]^\omega$, then $y = \{j(\alpha_n); n \in \mathbb{N}\}$, where $\alpha_n \in \hat{\kappa}$. But then $x := (n \mapsto \alpha_n)$ is a function from ω into $\hat{\kappa}$. By definition, we have that $j(x) = y$. Thus

$$j(f)(y) = j(f)(j(x)) = j(f(x)),$$

by elementarity. But that's false, since $\kappa \in \hat{\kappa}$, but there is no $\alpha \in \hat{\kappa}$ such that $\kappa = j(\alpha)$, as $\kappa = \text{crit}(j)$. Contradiction. □

8.4 Supercompactness

Definition 8.18 (Closed Under λ -Sequences). If λ is a cardinal and $\mathcal{M}$ an inner model, we say that $\mathcal{M}$ is **closed under λ -sequences** if $M^\lambda \subseteq M$.

Definition 8.19 (μ -Supercompact). We say an elementary embedding

$$j : V_\lambda \longrightarrow M,$$

is **μ -supercompact** if $\text{crit}(j) = \kappa$, $\mu < j(\kappa)$ and $\mathcal{M}$ is closed under μ -sequences.

There is an immediate relation between supercompactness and strength:

Lemma 8.20. If j is 2^κ -supercompact, where $\text{crit}(j) = \kappa$, then it is 2-strong.

Proof. If $X \in V_{\kappa+2}$, $X \subseteq V_{\kappa+1}$, hence

$$|X| \leq |V_{\kappa+1}| = 2^{|V_\kappa|} = 2^\kappa.$$

So since $V_{\kappa+1} \subseteq M$, we have that X is a 2^κ -sequence. $\square$

Theorem 8.21. If κ is measurable, and $\mathcal{M}$ is the ultrapower, then $M^\kappa \subseteq M$.

Proof. If $x \in M^\kappa$, then

$$\begin{aligned} x : \kappa &\longrightarrow M \\ x &\longmapsto ((f_\alpha); \alpha < \kappa), \end{aligned}$$

where f_α is the representative of $x(\alpha)$. Let $h : \kappa \rightarrow \kappa$ be such that $(h) = \kappa$. Define a function g with $\text{dom}(g) = \kappa$, that yields the right objects: take $g(\xi)$ to be the function g_ξ such that $\text{dom}(g_\xi) = h(\xi)$ and for all $\alpha \in \text{dom}(g_\xi)$, $g_\xi(\alpha) = f_\alpha(\xi)$.

We know in $\mathcal{M}$ by Los' Theorem 5.10 that

- (i) (g) is a function,
- (ii) $\text{dom}((g)) = (h) = \kappa$,
- (iii) If $a < \kappa$, $(g)(\alpha) = (f_\alpha)$.

So $(g) = x$, and therefore $x \in M$. $\square$

Definition 8.22 (Supercompact). A cardinal κ is called **μ -supercompact** if there is an embedding

$$j : V_\lambda \longrightarrow M,$$

such that j is μ -supercompact. Moreover, a cardinal κ is called **supercompact** if it is μ -supercompact for all μ .

Remark. We have

- (i) Measurable $\iff \kappa$ -supercompact,
- (ii) The ultrapower embedding is not κ^+ -supercompact (see ES#3.45).
- (iii) Kunen's Inconsistency Theorem 8.15 shows that no embedding with $\text{crit}(j) = \kappa$ is $\hat{\kappa}$ -supercompact.

9 Closing Loose Ends

Remark. Loose ends:

- (i) Large cardinals and GCH,
- (ii) Witness objects, what happens in reflection proofs.
- (iii) The annoying inaccessible above.

Remark. New large cardinal notions: Berkley/Toronto cardinals. He doesn't know what they are, would like to find out.

9.1 Large Cardinals and GCH

Theorem 9.1 (Scott). If κ is measurable and GCH holds below κ , i.e. $\forall \mu < \kappa, 2^\mu = \mu^+$, then $2^\kappa = \kappa^+$.

Proof. If GCH holds below κ , then by elementarity,

$$\begin{aligned} \mathcal{M} &\models \text{GCH holds below } j(\kappa), \\ \implies \mathcal{M} &\models 2^\kappa = \kappa^+. \end{aligned}$$

By our analysis, we know $V_{\kappa+1} \subseteq M$, so $\mathcal{P}(\kappa) \subseteq M$, and thus

$$\kappa^+ = (\kappa^+)^{\mathcal{M}} = (2^\kappa)^{\mathcal{M}} = 2^\kappa.$$

□

Remark. We'll improve this for supercompact cardinals:

Proposition 9.2. If κ is γ -supercompact and μ is such that $\mu \leq \gamma$ and GCH holds below κ , then $2^\mu = \mu^+$.

Proof. Essentially the same proof as Theorem 9.1 if $\gamma < j(\kappa)$. By our proof of Kunen's Inconsistency Theorem 8.15, we can WLOG assume that $\gamma < j(\kappa)$: we saw that for any j , j is not $\widehat{\kappa}$ -supercompact, so we know that there is some n such that $\gamma < \kappa_n$, i.e. the n -th iterate of j on κ .

So we can work with the n -th iterate of j instead which moves κ beyond γ . Then we give Scott's argument: because GCH holds below κ ,

$$\begin{aligned} \mathcal{M} &\models \text{GCH holds below } j(\kappa), \\ \implies \mathcal{M} &\models 2^\mu = \mu^+, \end{aligned}$$

and use γ -supercompactness to see that $\mathcal{P}(\mu)^{\mathcal{M}} = \mathcal{P}(\mu)$.

□

Remark. These results were historically important for their place within Gödel's programme to show validity of GCH by use of large cardinals (L.C. $\implies$ GCH).

9.2 Witness Objects

Remark. Recall the Fundamental Theorem [7.12](#).

Theorem 9.3 (Fundamental). TFAE:

- (i) $\exists j : V_\lambda \rightarrow M$ elementary $\text{crit}(j) = \kappa$,
- (ii) $\exists U$, κ -complete non-trivial u.f. on κ .

Remark. Here we have two existential statements, one about j which is an object of $V_{\lambda+1}$, and U is an object of $V_{\kappa+2}$.

Remark. The power of the fundamental theorem is that the elementarity is witnessed by an object in $V_{\kappa+2}$. This is used in **reflection arguments**.

Example 9.4. If κ is 2-strong, then κ is the κ -th measurable.

This uses the fact that measurability is witnessed by something in $V_{\kappa+2}$, which is then preserved by the (2-strong) embedding.

Question. What are the witness objects for supercompactness/strength/strong compactness?

9.2.1 For Supercompact Cardinals

Remark. See [\[Kan03a, Theorem 22.7\]](#), due to Solovay and Reinhardt.

Notation. We write

$$\mathcal{P}_\kappa\gamma := \{X \in \mathcal{P}(\gamma); |X| \leq \kappa\}.$$

Theorem 9.5 (Solovay–Reinhardt). If $\kappa \leq \gamma$, κ is γ -supercompact iff there is an ultrafilter over $\mathcal{P}_\kappa\gamma$ satisfying some combinatorial condition called *normality*.

Question. Where does an ultrafilter on $\mathcal{P}_\kappa\gamma$ live?

Answer. In $V_{\gamma+2}$!

Remark. In this way, Theorem [9.5](#) is reduced from a statement about elementary to a statement about objects in $V_{\gamma+2}$.

Exercise. If κ is supercompact and $\lambda > \kappa$ is inaccessible, then $V_\lambda \models \kappa$ is supercompact. Moreover, if

$$\text{Cons}(\text{ZFC} + \exists \kappa (\kappa \text{ is supercompact})),$$

then

$$\text{Cons}(\text{ZFC} + \exists \kappa (\kappa \text{ is supercompact} \wedge \neg \exists \lambda (\lambda > \kappa \wedge \lambda \text{ is inaccessible}))).$$

Remark. This exercise allows us to show that the assumption of extra inaccessibles is stronger.

9.2.2 For Strongly Compact Cardinals

Definition 9.6. A cardinal κ is called γ -**compact** if there is a *fine ultrafilter* on $\mathcal{P}_\kappa \gamma$.

Theorem 9.7 ([Kan03a, Theorem 22.17]). If $\kappa \leq \gamma$, the following are equivalent:

- (i) κ is γ -compact,
- (ii) There is a $j : V \prec M$ with $\text{crit}(j) = \kappa$ such that for any $X \subseteq M$ with $|X| \leq \gamma$, there is a $Y \in M$ such that $Y \supseteq X$ and $M \models |Y| < j(\kappa)$,
- (iii) For any set S , every κ -complete filter over S generated by at most $|\gamma|$ sets can be extended to a κ -complete ultrafilter over S .

Remark. Asking for (iii) for every γ gives the reverse direction of the Keisler-Tarski Theorem 5.6: if every κ -complete filter over any set S can be extended to a κ -complete ultrafilter over S , then κ is strongly compact. See [Kan03a, Proposition 4.1].

9.2.3 For Strong Cardinals

Remark. Much more involved than previous.

Definition 9.8 (Extender). A (κ, γ) -**extender** is a family of ultrafilters

$$\{E_a; a \subseteq \gamma \text{ finite}\},$$

with certain cohesion properties.

Remark. These objects are very important for modern research in large cardinals.

Remark. These extenders can then be used as witness objects for strong embeddings. See [Kan03a, Section 26, *Extenders*].

9.3 The Annoying Inaccessible

Theorem 9.9 (Fundamental). TFAE:

- (i) $\exists j : V_{\lambda+1} \rightarrow M$ elem. crit(j) = κ ,
- (ii) $\exists U$ κ -complete non-trivial u.f. on κ .

Remark. In order to remove the inaccessible, we'd like to be able to express in a model of set theory that there is an elementary embedding. But j is an object in $V_{\kappa+1}$!

If we are working in a fixed model $\mathcal{V} \models \text{ZFC}$, then both j and M would be proper classes and therefore we cannot quantify over them.

Question. How to handle classes?

Remark. If $\mathcal{V}$ is a model of set theory that is a *set* in our meta-set theory, then its classes are just $\mathcal{P}(V)$, where the power-set is taken in the meta-set theory.

Many classes that matter in practice are definable, e.g. ordinals. If φ is a formula with $n + 1$ free variables and $\vec{x} \in V^n$, then we can apply meta-separation to get

$$C_{\varphi, \vec{x}} := \{x \in V; \mathcal{V} \models \varphi(x, \vec{x})\}.$$

But there are $|V|$ definable classes over V but $2^{|V|}$ classes over V !

Question. Can the fundamental theorem be reformulated as a meta-theorem about definable classes?

Theorem 9.10 (Meta-Fundamental). TFAE:

- (i) $\exists \varphi, \vec{x}$ such that $C_{\varphi, \vec{x}}$ is an elementary embedding with critical point κ ,
- (ii) $\exists U$, κ -complete non-trivial over κ .

Remark. The elementary embedding uses infinitely many formulas. Alternatively, this can be done in a first-order way in some standard class theory, e.g. von Neumann-Bernays-Gödel (NBG) class theory.

Proof. (i) $\implies$ (ii) direction carries over, replacing all references to j by appropriate terms involving φ and $\vec{x}$.

(ii) $\implies$ (i) direction: we'd like a formula using U as a parameter such that

$$\begin{aligned} x \in M &\iff \Phi(x) \\ &\iff x \text{ is image of Mostowski collapse of some } [f]_U, f : \kappa \rightarrow V, \end{aligned}$$

where $f \in V$ by Replacement. However, the equivalence class $[f]_U$ is a proper class and not an element of V : if f is any function and $x \in V$ is any set then

$$f_x : \alpha \mapsto \begin{cases} x & \text{if } \alpha = 0 \\ f(\alpha) & \text{if } \alpha \neq 0 \end{cases}$$

satisfies $f_x \sim_U f$, as U is non-principal, but the ranks of the f_x are unbounded in V .

We now use **Scott's Trick**: consider the class

$$[f]_U \subseteq V,$$

and find α minimal such that $\text{scott}(f) = [f]_U \cap V_\alpha \neq \emptyset$.

Clearly, if $f \not\sim_U g$, then

$$\text{scott}(f) \cap \text{scott}(g) = \emptyset.$$

Our formula Φ is

$$\Phi(x) \iff x \text{ is the Mostowski collapse of some } \text{scott}(f) \text{ for some } f : \kappa \rightarrow V.$$

□

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