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Quantum Field Theory

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0 Introduction

Quantum Field Theory is a marriage of quantum mechanics and special relativity which provides the mathematical framework for describing the interactions of elementary particles. This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalars, (Dirac) spinors, and gauge fields. The relativistic covariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem. The quantisation of free fields is developed using Hamiltonian methods in terms of operators which create and annihilate particles and anti-particles. The Fock space of quantum physical states and the associated realisation of particle statistics is described. Interacting field theory is developed next using Dyson's formula for the scattering amplitude. Perturbation theory is formulated by applying Wick's theorem and Feynman diagrams are introduced as an efficient way of organising the resulting calculations. Physically measurable quantities such as decay rates and cross-sections are defined. Spinors and the Dirac equation are explored in detail, along with parity and chirality. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions. We review the relativistic formulation of Maxwell's equations formulating a corresponding action principle. The significance of gauge invariance is discussed. Lorentz gauge quantisation of the electromagnetic field is accomplished by imposing the Gupta-Bleuler condition. Finally, quantum electrodynamics (QED) is developed. Interactions between photons and charged matter are introduced via the principle of minimal coupling. We discuss the calculation of tree-level scattering amplitudes in QED.

Main references are [Pes95, Ton06, Zee10, LB14].

Lecture 1

1 Why do we need QFT?

Culmination of ideas of relativity and QM. QED agreement between theory and experiments of 10 fig. Very much WIP. No rigorous mathematical defn for what a QFT should be. Related to having no general procedure/generally applicable methods to turn a QFT into consequences for experiments.

We can sometimes do perturbative expansion in terms of Feynman diagrams, actually a divergent series, leading to renormalisation, feature more than bug. Those series in themselves do not provide a rigorous definition of QFTs. We work only here at lowest levels, "tree-level QFTs". By the end of it, we'll be able to calculate scattering amplitudes at leading order.

First example sheet out.

But why do we need QFT?

Consider the naive story of attraction between positive and negative charges in electrostatics. Coulomb's force. We need to change it in two directions:

c Pass to electrodynamics, describe the force in terms of fields, get a vector depending on time, get also time-dependent vector-valued magnetic fields. Features of locality, no action at a distance, and Lorentz invariance, already embedded in the theory, as get waves that propagate at the speed of light. Can combine E, B into a Lorentz tensor, which is Lorentz invariant.

$\hbar$ Pass to quantum mechanics. Features of "discreteness/quantized theory" and probabilistic theory.

If we combine both ingredients, we'll need QFT. New qualitative features, such as particle creation, provided that we preserve the overall energy. Photon obeying

$E_\sigma \geq 2m_e c^2$, we can create a particle-antiparticle pair (not actually physically allowed). Number of particles not a conserve quantity.

Aside from spins, we cannot distinguish elementary particles. Bose/Fermi statistics, get (anti)symmetrization of wave function. In QM, this has to be put in by hand, such as Pauli-exclusion principle. We want this to be produced naturally. The implication of spin-statistics theorem was an early validation test.

Typical experiment involving beams of elementary particles at high energy. Idealized by some process which takes some initial state $|i\rangle$, get final state $|f\rangle$ with potentially more particles. Our goal will be to compute the probability $p_{i \rightarrow f} = |A_{i \rightarrow f}|^2$, i.e. mod-squared of amplitude, which is just a complex number. When we success in such a calculation, we need to require that the theory is unitary, i.e. $\sum_f p_{i \rightarrow f} = 1$, i.e. the S matrix is unitary. Moreover, if we feed Lorentz invariant input, we want a Lorentz invariant output, i.e. our theory must generate Lorentzian-covariant outputs. In practice, these requirements are very restrictive.

What is a QFT?

We cannot start from QM. We start from Maxwell's classical field theory $\varphi(\vec{x}, t)$. This classical field obeys Lorentzian invariance. We will describe the dynamics first in terms of Lagrangians and then Hamiltonians of the theory. At this point, there is a canonical procedure to a classical phase space to a quantum theory with Hermitian operators. We will follow a similar procedure to get the quantum field operator $\hat{\varphi}(\vec{x}, t)$, as a result of applying the canonical quantization.

This is hard as we are really quantizing an infinite-degrees of freedom, which will be our source of UV divergences. Will need distribution theory.

Linear field equations will be our starting point. We then look for the energy eigenstates of the Hamiltonian $\hat{H}[\hat{\varphi}, \hat{\pi}]$, which match the requirements **multi-particle states**. We now compute scattering amplitudes by

$$\mathcal{A}_{i \rightarrow f} = \langle f | e^{i\hat{H}T} | i \rangle,$$

with the time-evolution operator. But actually getting finite consistent answers from this is very limiting. For QED, we have the options given for us by the theory, which makes it easy to identify those choices of gauge which give physical theories.

Outline:

- (i) Classical Field Theory,
- (ii) Free QFT,
- (iii) Interacting QFT, treating perturbatively in the scalar field/spin-0 particles,
- (iv) Fermions, gauge fields,
- (v) QED

Closest reference to course material is [Pes95].

2 Preliminaries

2.1 Units

Three fundamental constants: $c \sim LT^{-1}$, $\hbar \sim L^2MT^{-1}$, $G \sim L^3M^{-1}T^{-2}$, linearly independent sets of quantities.

Definition 2.1. We define the **natural units** to be given by $c = \hbar = 1$.

Using natural units, all quantities scale with some power of mass. In terms of notation:

$$X \sim M^\delta,$$

where δ is the **scaling dimension**. For a quantity X , we denote $[X] := \delta$.

Example 2.2. In natural units, $[E] = +1, [x] = -1$.

2.2 Relativity

We work in flat Minkowski space $\mathbb{R}^{3,1}$. Metric tensor $\eta_{\mu r} = \text{diag}\{+1, -1, -1, -1\}$, mostly-minus convention. The coordinates $x^\mu = (t, \vec{x})$.

We also adopt the summation convention

$$x_\mu := \eta_{\mu r} x^r,$$

where summation over r is implied. Inverse metric $\eta^{\mu r}$, which gets us

$$\eta^{\mu l} \eta_{lr} = \delta_r^\mu.$$

3 Classical Field Theory

Theory of scalar field: assignment a real number at each point in time.

$$\begin{aligned} \varphi : \mathbb{R}^{3,1} &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi = \varphi(x) = \varphi(t, \vec{x}). \end{aligned}$$

What kinds of functions do we allow? What are the boundary conditions on Minkowski space? Smooth or continuous? Generally, if you start with smooth inputs, the QFT propagates it to a smooth outputs.

If we want to think of fields in general, φ maps from some spacetime to a "field space". Field spaces could be for example *spin spaces*, we want to be able to carry a representation of the Lorentz group. In generalisations, we could have some manifold as a field space.

In the other direction, we can replace spacetime with something curved.

Lecture 2

Scalar field

$$\varphi : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}.$$

Write $\varphi(x) = \varphi(t, \vec{x})$.

Main property of a scalar field: how it transforms under Lorentz transformation.

Remark. The **Minkowski metric** is invariant under LT.

Recall that LT are

$$x^\mu \longrightarrow (x')^\mu = \Lambda_\nu^\mu x^\nu,$$

just think of Λ as a 4x4 matrix which obey

$$\Lambda_\sigma^\mu \Lambda_\tau^\nu \eta^{\sigma\tau} = \eta^{\mu\nu},$$

defining equation of LTs. In matrix notation

$$\Lambda \cdot \eta \cdot \Lambda^\top = \eta.$$

Definition 3.1. We say an LT is **proper** if $\det \Lambda = +1$.

Proposition 3.2. LTs form a group under composition/matrix multiplication.

Definition 3.3. We call the subgroup of proper LTs the **Lorentz group** $G_L = SO(3, 1)$.

Remark. This is a Lie group, i.e. a group that is also a manifold in a compatible way. We'll leave most such technicalities to SFT.

Question. How does a scalar field transform under Lorentz transformation, i.e. as we pass from one inertial frame to another?

$$\varphi(x) \xrightarrow{\Lambda \in G_L} \varphi'(x) := \varphi(\Lambda^{-1} \cdot x).$$

We call this the **active convention for transformations**. See discussion in PS/Tong on active vs passive conventions for transformations.

Let us make this more explicit:

$$(\Lambda^{-1} \cdot x)^\mu = (\Lambda^{-1})^\mu_\nu x^\nu,$$

where $\Lambda^{-1} \in SO(3, 1)$ such that $(\Lambda^{-1})^\mu_l \Lambda^l_\nu = \delta^\mu_\nu$.

3.1 Field Constructions

Before we go into Lagrangians, let us construct more fields out of φ . We know we will be dealing with differential equations governing the laws of motion, so let us consider derivatives.

Definition 3.4. We define the **spacetime derivative field** by

$$\partial_\mu \varphi(x) := \frac{\partial \varphi(x)}{\partial x^\mu}.$$

We can work out the transformation law as:

$$\begin{aligned} \partial_\mu \varphi(x) \xrightarrow{\Lambda \in G_L} (\partial_\mu \varphi)'(x) &= \frac{\partial}{\partial x^\mu} \varphi(\Lambda^{-1} \cdot x) \\ &= (\Lambda^{-1})^\nu_\mu (\partial_\nu \varphi)(\Lambda^{-1} x), \end{aligned}$$

by chain rule, But due to the factor in front, the derivative is not a scalar field!

Definition 3.5. A (4-)vector field is a map

$$V : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}^{3,1},$$

which transforms as

$$V^\mu(x) \xrightarrow{\Lambda \in G_L} \Lambda^\mu_\nu V^\nu(\Lambda^{-1} \cdot x).$$

Remark. Note that here it is just a function, but in general, with non-flat spacetime, we need sections of tangent bundles.

This gives us the generic notion that transforms as the representations of the Lorentz group, the general concept of whom is the tensor field. The fact that it is a field will mean that we transform the spacetime argument.

Exercise. Try to

- (i) Show that

$$\partial^\mu \varphi(x) := \mu^{\mu\nu} \partial_\nu \varphi(x),$$

is a vector field. [Hint: use defining property of Lorentz transformation.] Note that raised index means just acting with the Lorentz tensor.

- (ii) Prove that the following field by contraction is a scalar field

$$K(x) = \partial_\mu \varphi(x) \partial^\mu \varphi(x).$$

Remark. We want (ii) to construct Lagrangian fields which are scalar fields.

3.2 Lagrangians

Idea. We want to start thinking of dynamics from the point of view of Lagrangian mechanics.

Recall that in particle mechanics there are several advantages to using Lagrangians:

- (i) We can transform ?
- (ii) Lagrangian are invariant under the symmetries of the theory, which is not true for the EOM, which transform covariantly. This will take the form of Noether's Thm.
- (iii) We can use Lagrangians to pass to Hamiltonians, which are the correct setting to perform quantizations.

Review of Lagrangians for non-relativistic particle of $m = 1$ in potential in 1d.

Position is just $q = q(t)$. Can define dyn-sys by either giving EOM or giving a Lagrangian.

$$\begin{aligned} L(t) &= L(q(t), \dot{q}(t)) \\ &= \frac{1}{2} \dot{q}^2(t) - V(q). \end{aligned}$$

Definition 3.6. The **action** in time interval $[t_i, t_f]$ is

$$S[q] = \int_{t_i}^{t_f} dt L(q(t), \dot{q}(t)).$$

Remark. The action is a functional, which is why we write $[-]$. It depends on the whole path. It is overkill in classical dynamics, but it introduces us here to the ideas of QM.

The Principle of Least Action

(i) Vary path of particle

$$q(t) \rightarrow q(t) + \delta q(t),$$

by variations with quick endpoints, i.e. $\delta q(t_i) = \delta q(t_f) = 0$.

(ii) Demand stationary action

$$\delta S = 0,$$

for all such $\delta q(t) = 0$.

Then the principle of least of least action tells us this will be the case iff the Euler-Lagrange Equations hold. In our case, this reduces

$$\ddot{q} = -\frac{\partial V}{\partial q}.$$

This is overkill for Newtonian mechanics, but will be vital for our QFT picture. To have a Lagrangian for φ a scalar field, we need

- (i) A Lorentz invariant action,
- (ii) It is local, which follows naturally when we involve derivatives. In practice we mean interactions involving position and its derivative must share the given point.
- (iii) Renormalizability, which determines whether the scalar theory makes sense. We phrase this as $[\text{Lagrangian density/terms that occur in } \mathcal{L}] \leq 4$.

Let us write a Lagrangian for $\varphi(x) = \varphi(t, \vec{x})$,

$$L(t) = "L[\varphi]" = \int d^3x \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)),$$

by locality. This ensures that when we apply the principle of LA we will get an PDE.

Definition 3.7. We call $\mathcal{L}$ the **Lagrangian density**.

Question. Are locality algebras a sensible idea as where these Lagrangians come from?

Definition 3.8. We define the **action** as

$$S_{t_i, t_f}[\varphi] = \int_{t_i}^{t_f} L[\varphi] = \int_{t_i}^{t_f} \int d^3\Lambda.$$

This makes sense when there is an initial and final time, such as when we're interested in scattering. But we want to think in generality to infinite time intervals, letting $t_i \rightarrow -\infty, t_f \rightarrow \infty$. This gives the action a much nicer form as

$$S[\varphi] = \int_{\mathbb{R}^{3,1}} d^4x \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)).$$

We want the action to be Lorentz invariant, ideally without imposing that $\mathcal{L}$ be Lorentz invariant itself. We will see that it suffices for $\mathcal{L}$ to be a scalar field.

LT $\Lambda \in SO(3,1)$, require $\mathcal{L}$ is a scalar field, i.e.

$$\mathcal{L}(x) \xrightarrow{\Lambda} \mathcal{L}(\Lambda^{-1}x).$$

Then

$$S = \int d^4x \mathcal{L}(x) \xrightarrow{\Lambda} S' = \int d^4x \mathcal{L}(\Lambda^{-1} \cdot x).$$

Change variables and define $y^\mu = (\Lambda^{-1})^\mu_\nu x^\nu$. We pick up a Jacobian, but that is fine, as it is just $\det \Lambda = +1$. Thus

$$S' = \int_{\mathbb{R}^{3,1}} y \mathcal{L}(y) = S.$$

Lecture 3

Corrections:

Remark. A **contravariant** 4-vector is one with a raised index x^μ . A **covariant** 4-vector is one with a lowered index x_μ .

Remark. Most general $\mathcal{L}$

- (i) Lorentz invariant,
- (ii) ≤ 2 time derivatives.

We could have

$$\mathcal{L} = \frac{1}{2} F(\varphi) \partial_\mu \varphi \partial^\mu \varphi - V(\varphi),$$

which is a reasonable classical field theory, but in the quantum setting, we'll have to add a (iii) requirement of $F = 1$ for renormalization. This gets us

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi).$$

In the general case with no assumption on V , e.g. being quadratic, we don't get Klein-Gordon but

$$\partial_\mu \partial^\mu \varphi + V'(\varphi) = 0,$$

and if V is not quadratic, this means we have a non-linear PDE, which is hard in the general case. We can only find wave-like solutions in small perturbations. If we try to get wave-packets solutions, they don't stay local and linearity also breaks, so no superpositions.

Moreover, we get other funky solutions like non-linear waves, e.g. solitons, they have a role which will be explored in ET23 "Solitons".

If we proceed to quantize this, we'll find a theory with interacting particles, which is hard unless we're using perturbation theory.

3.3 Maxwell's Theory as an Example

4-vector potential regarded as a 4-vector field:

$$A^\mu(x) = A^\mu(\vec{x}, t) = (\varphi, \vec{A}),$$

scalar and vector potential.

Thinking of it as a 4-vector field in Minkowski spacetime, we want to consider a behaviour under standard LT Λ . Then A^μ transforms as

$$A^\mu(x) \xrightarrow{\Lambda} \Lambda^\mu_\nu A^\nu(\Lambda^{-1} \cdot x). \quad (1)$$

The EM field live in field-strength/Maxwell tensor,

$$F^{\mu\nu}(x) := \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \quad (2)$$

Corresponding transformation rule under LT:

$$F^{\mu\nu} \xrightarrow{\Lambda} \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}(\Lambda^{-1} \cdot x). \quad (3)$$

This follows from Equation 2 by chain rule.

Moreover, $F^{\mu\nu}$ is invariant under gauge transformation, another transformation of the field given by

$$A^\mu(x) \xrightarrow{\lambda} A^\mu(x) + \partial^\mu \lambda(x),$$

where $\lambda : \mathbb{R}^{3,1} \rightarrow \mathbb{R}$ is arbitrary, potentially with some decay at infinity.

Since by equation 2, F depends on A , it will satisfy the Bianchi identity, which may also be written using the alternating tensor ε

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

| **Exercise.** Check this.

This actually encodes two of Maxwell's equations, once we re-express F in terms of electric/magnetic potential.

3.3.1 The Maxwell Lagrangian

We need a gauge-invariant, Lorentz-invariant Lagrangian-density with at most two time derivatives. This basically reduces us to

$$\Lambda = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

which is Lorentz invariant because we were careful with indexes and gauge-invariant because we expressed this in terms of F and not A .

To check EOM, we need to rewrite this in terms of the 4-vector field, because we need to vary the field. This is hard because this obeys the Bianchi identity. The 4-vector field is unconstrained however, so we can vary that.

Expanding out

$$\begin{aligned} \Lambda &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{2} (\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2} (\partial_\mu A^\mu)^\nu + \text{total derivative.} \end{aligned}$$

This gets us

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) = 0$$

In our case, we can evaluate the derivatives explicitly:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= 0, \\ \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} &= -\partial^\mu A^\nu + (\partial_t A^t) \eta^{\mu\nu}, \end{aligned}$$

hence EL becomes

$$\partial_\mu F^{\mu\nu} = 0 \tag{4}$$

This yields the remaining two Maxwell's equations, and it will give us an example for QED. Quantizing Maxwell's theory will give us a theory of interacting spin-1 particles.

3.4 Symmetries

Definition 3.9. A **symmetry** is variation of fields which leaves the **action** invariant.

The symmetries of a theory get imprinted on the observables in several ways:

- (i) Through conservation laws/conserved charges, by continuous symmetries. This constraints which processes our theory allows and will be content of Noether's Theorem.
- (ii) The right way to think of symmetries is with representations of non-abelian (Lie) groups, e.g. eight-fold way of charges of hadrons.

A symmetry restricts the form of the Lagrangian, e.g. demanding that it is Lorentz-invariant heavily restrict us. The most powerful form of this in a gauge theory is that the entire form of the Lagrangian is dictated by gauge invariance, e.g. in the Standard Model.

This effect is powerful already for Lorentz invariance, but is even stronger with gauge-invariance. It effectively reduces the problem of finding Lagrangians, e.g. for the Standard Model, to picking the correct gauge groups and their representations. We will see that gauge invariance is essential for renormalizability.

Want to distinguish global/spacetime symmetries from gauge symmetries, which leave also the other quantities of interest unchanged.

Let us focus on symmetries of the scalar field theory. The spacetime symmetries that occur are:

- (i) Translation

$$\begin{aligned} x^\mu &\rightarrow x^\mu + c^\mu, \\ \varphi(x) &\rightarrow \varphi(x - c), \end{aligned}$$

where $c \in \mathbb{R}^{3,1}$. This is automatic by using Lagrangian densities.

- (ii) Lorentz

$$\begin{aligned} x^\mu &\rightarrow \Lambda^\mu_\nu x^\nu, \\ \varphi(x) &\rightarrow \varphi(\Lambda^{-1} \cdot x), \end{aligned}$$

$$\Lambda \in G_L.$$

- (iii) Under the widely applicable assumptions that $m = 0, V = 0$ and scaling as $\alpha\varphi^4$, we get dilatation/scale transformation

$$\begin{aligned}x^\mu &\rightarrow \lambda x^\mu, \\ \varphi(x) &\rightarrow \varphi^{-\Delta}(\lambda^{-1} \cdot x),\end{aligned}$$

where $\Delta = [\varphi]$ and $\lambda \in \mathbb{R}^+$. This often becomes conformal field symmetry.

We also have internal symmetries. E.g. electric charge, flavour, colour (a gauge theory).

We'll stick with a scalar field, but generalise it to be complex.

$$\psi : \mathbb{R}^{3,1} \rightarrow \mathbb{C}.$$

We write down a Lagrangian for ψ , which we want to be real, as we'll want a real Hamiltonian:

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2),$$

where $|\psi|^2$ is because we want the symmetry under phase rotation

$$\begin{aligned}\psi(x) &\rightarrow \psi'(x) = e^{i\alpha} \psi(x), \\ \psi^* &\rightarrow \psi^{*\prime}(x) = e^{-i\alpha} \psi^*(x),\end{aligned}$$

for $\alpha \in [0, 2\pi)$, which means that both $\mathcal{L}$ and S are invariant.

Generally, continuous symmetries yield matrix/linear Lie groups.

Example 3.10. L.T. $x^\mu \rightarrow \Lambda^\mu_\nu z^\nu$, a 4x4 matrix. Then

$$G_L = \{\Lambda \in \text{Mat}_4(\mathbb{R}) | \Lambda \cdot \eta \cdot \Lambda^\top = \eta, \det \Lambda = 1\} = SO(3, 1).$$

Remark. $SO(3,1)$ Lorentz, but also have translations and rotations. Translations plus translations gives the Poincaré group, which we can enlarge to $SO(4,2)$ which is the conformal group.

3.5 Finite vs Infinitesimal Transformations

Thinking of symmetry transformations as elements of a group, group near $g \in G$ "near the identity" can be written in exponential form as

$$g = \exp(\alpha X),$$

with $\alpha \in \mathbb{R}$, X a matrix. But what does $\exp$ mean with matrices? The matrix $X \in L(G)$ belongs to the Lie algebra of G . The exponential of a matrix is defined by

$$\exp(\alpha X) := \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha X)^n,$$

which for $\alpha \ll 1$ is approximated by

$$1 + \alpha X + O(\alpha^2),$$

often convenient to work up to 1st order.

Example 3.11. For our

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2),$$

the finite transformation

$$\psi(x) \rightarrow g \cdot \psi(x),$$

where $g = \exp(i\alpha) \in G \sim U(1)$.

An infinitesimal transformation

$$\psi(x) \rightarrow \psi(x) + \delta\psi(x),$$

is then to first order

$$\delta\psi(x) = i\alpha\psi(x),$$

which will be sufficient for most purposes.

Lecture 4

For LT $\Lambda \in G_L = SO(3,1)$, in a neighbourhood of the identity, we can write

$$\Lambda \simeq \exp(s\Omega),$$

where the matrix Ω belongs to the Lie algebra of $SO(3,1)$. To be fully explicit, this means that we can expand

$$\Lambda_\nu^\mu = \delta_\nu^\mu + s\Omega_\nu^\mu + O(s^2),$$

imposing the defining property of LTs w.r.t. the metric tensor, we get

$$\begin{aligned} (\delta_\nu^\mu + s\Omega_\nu^\mu)\eta^{\alpha\beta} \times \\ (\delta_\beta^\nu + s\Omega_\beta^\nu) = \eta^{\mu\alpha}, \end{aligned}$$

hence at first order we get

$$\begin{aligned} s\Omega^{\mu\nu} + s\Omega^{\nu\mu} &= 0 \\ \implies \Omega_{\mu\nu} &= -\Omega_{\nu\mu}. \end{aligned}$$

We use this to note that this gets us exactly $\frac{1}{2}3 \cdot 4 = 6 = 3 + 3$ degrees of freedom, split across 3 for rotations and 3 for boosts.

Note that $\det \Lambda = 1$ is automatic near the identity, so we need not impose it.

The scalar field now transforms as

$$\varphi(x) \rightarrow \varphi(\Lambda^{-1} \cdot x),$$

hence

$$(\Lambda^{-1})_\nu^\mu = \delta_\nu^\mu - s\Omega_\nu^\mu + O(s^2).$$

Hence substituting back we get

$$\begin{aligned} \varphi(\Lambda^{-1} \cdot x) &\simeq \varphi(x - s\Omega \cdot x) \\ &\simeq \varphi(x) - s\Omega_\nu^\mu x^\nu \partial_\mu \varphi(x) + O(s^2), \end{aligned}$$

so in linear order, the LT has the effect of producing a linear shift in the field. This helps us identify the variation in the field, with the infinitesimal L.T.

$$\varphi(x) \rightarrow \varphi(x) + \delta\varphi(x),$$

where

$$\delta\varphi(x) = -s\Omega_\nu^\mu x^\nu \partial_\mu \varphi(x). \quad (5)$$

3.6 General Story for Symmetries

Useful to understand the internal symmetries of the standard model. A good source of internal symmetries is to have multi-component fields, i.e. the field space being a vector space, be it real or complex. Let us consider the case of complex fields, which live in $\mathbb{C}^n$.

Warning. Note that we are really considering scalar fields with multi-components, not vector fields!

Denote

$$\vec{\psi} : \mathbb{R}^{3,1} \longrightarrow \mathbb{C}^n,$$

with inner product

$$(\vec{\psi}_1, \vec{\psi}_2) = \vec{\psi}_1^\dagger \cdot \vec{\psi}_2.$$

Idea of a group representations: a Lie group can act in different ways on vector fields of various dimensions, as witnessed by the datum of a representation.

Definition 3.12. An n -dimensional representation D of a symmetry group G

$$D : G \rightarrow \text{Mat}_n(\mathbb{C}),$$

which preserves group multiplication, i.e.

$$D(g_1 \cdot g_2) = D(g_1) \cdot D(g_2).$$

In physics, we really want to focus on unitary representations:

Definition 3.13. We say a representation D of G is **unitary** if

$$\forall g \in G. D(g)^{-1} = D(g)^\dagger.$$

If we have a unitary representation, we can write down a Lagrangian that has exactly that symmetry:

$$\mathcal{L} = (\partial_\mu \vec{\psi}, \partial^\mu \vec{\psi}) - V((\vec{\psi}, \vec{\psi})),$$

is invariant under

$$\vec{\psi}(x) \rightarrow \vec{\psi}'(x) = D(g) \cdot \vec{\psi}(x),$$

for all $g \in G$. To see this, it suffices to show that

$$\begin{aligned} (\vec{\psi}, \vec{\psi}) &\rightarrow (D(g) \cdot \vec{\psi}, D(g) \cdot \vec{\psi}) \\ &= (D(g)^\dagger \cdot D(g) \cdot \vec{\psi}, \vec{\psi}) \\ &= (\vec{\psi}, \vec{\psi}), \end{aligned}$$

by unitarity. This works in generality!

The simplest possibility is that $G = U(n)$, i.e. it is itself the group of unitary $n \times n$ matrices. In this case we can take the **fundamental representation**

$$\forall g \in G. D_F(g) = g.$$

This gives us a global symmetry of $U(n)$.

3.7 Noether's Theorem

Idea. Continuous symmetry give conserved currents.

We want to describe continuous terms in terms of variations of the field

$$\begin{aligned}\varphi(x) &\rightarrow \varphi(x) + \delta\varphi(x), \\ \delta\varphi(x) &= X(\varphi(x), \partial\varphi(x), \dots).\end{aligned}$$

In the case of a Lorentz transformation, we can take $\Lambda = \exp(s\Omega)$.

$$X = -s\Omega_{\nu}^{\mu}x^{\nu}\partial_{\mu}\varphi(x).$$

In the even simpler case of a translation

$$\varphi(x) \rightarrow \varphi(x - \varepsilon),$$

for some constant 4-vector ε^{μ} . Expanding to first order

$$X = -\varepsilon^{\mu}\partial_{\mu}\varphi(x).$$

As we have seen, in general, the Lagrangian density itself is not invariant under these symmetries. For the variation of $\mathcal{L}(x)$, we get under LT that, since $\mathcal{L}(x)$ is a scalar field,

$$\begin{aligned}\implies \delta\mathcal{L}(x) &= -s\Omega_{\nu}^{\mu}x^{\nu}\partial_{\mu}\mathcal{L}(x) \\ &= -s\Omega_{\nu}^{\mu}\partial_{\mu}(x^{\nu}\mathcal{L}(x))\end{aligned}$$

since $\Omega_{\nu}^{\mu} = \eta^{\mu\nu}\Omega_{\mu\nu} = 0$ by contraction of symmetric matrix with an anti-symmetric one. Hence we can express $\delta\mathcal{L}$ in terms of a **total derivative**.

For translations, $\mathcal{L}(x) \rightarrow \mathcal{L}(x - \varepsilon)$, we can see that

$$\delta\mathcal{L}(x) = -\varepsilon^{\mu}\partial_{\mu}\mathcal{L}(x),$$

and we get again a total derivative.

We'll see that this matches a general picture. Assume for the moment that for a general symmetry,

$$\delta\mathcal{L}(x) = \partial_{\mu}F^{\mu}, \tag{6}$$

for some $F^{\mu} = F^{\mu}(\varphi(x), \partial\varphi(x), \dots)$.

Now the variation of the action is given by

$$\begin{aligned}\partial S &= \int_{\mathbb{R}} d^4x \delta\mathcal{L} = \int_{\mathbb{R}} d^4x \partial_{\mu}F^{\mu} \\ &= \int_{\partial\mathbb{R}^{3,1}} dS_{\mu}F^{\mu} \quad (\text{Stoke's}),\end{aligned}$$

where s_{μ} is a surface term, though not necessarily zero, as some fields may fail to decay to zero at infinity. This is actually what we want to be a symmetry, it need not vanish at infinity, but it is allowed to be a surface term. This is what will allow us to have non-trivial symmetries which lead to currents. Note that surface terms do not affect the EOM by Euler-Lagrange.

Other half of the story for Noether's Theorem: **conserved currents**. Hence

$$j^{\mu}(x) = j^{\mu}(\varphi(x), \partial\varphi(x), \dots),$$

which we study for scalar fields, but has straightforward generalisations for more general fields. Hence the conservation equations are

$$\partial_\mu j^\mu = 0. \quad (7)$$

But this need only be the case when $\varphi(x)$ obeys the Euler-Lagrange Equations!

Let us explore the consequences of conserved current.

$$j^\mu(x) = j^\mu(\vec{x}, t) = (j^0(\vec{x}, t), \vec{J}(\vec{x}, t)),$$

which will refer to as **charge density** and **current density** respectively.

Definition 3.14. We'll define **total charge** in any region of space $V \subseteq \mathbb{R}^3$

$$Q_V(t) = \int_V d^3x j^0(\vec{x}, t).$$

Since by Equation 7 we have $\partial_\mu j^\mu = 0$, we get

$$\frac{\partial}{\partial t} j_0(\vec{x}, t) + \vec{\nabla} \cdot \vec{J}(\vec{x}, t) = 0, \quad (8)$$

and thus

$$\begin{aligned} \frac{d}{dt} Q_V(t) &= \int_V d^3x \frac{\partial}{\partial t} j^0(\vec{x}, t) \\ &= - \int_V d^3x \vec{\nabla} \cdot \vec{J}(\vec{x}, t) \quad (8) \\ &= - \int_{\partial V} d\vec{S} \cdot \vec{J}(\vec{x}, t) \quad (\text{Stoke's}), \end{aligned}$$

this gives the **flux through** V . We now consider the case that $V = \mathbb{R}^3$ and $\varphi(x) \rightarrow 0$ as $|\vec{x}| \rightarrow \infty$. The total charge then

$$Q(t) = \int_{\mathbb{R}^3} j^0(\vec{x}, t) d^3x,$$

where

$$\frac{d}{dt} Q(t) = - \int_{\partial \mathbb{R}^3} d\vec{S} \cdot \vec{J}(\vec{x}, t) = 0.$$

This is a familiar picture, but will be immensely useful in our quantum picture. For example, if we have a bunch of particles that obey conservation of charge, even if their total number changes, this constraint will remain on the results of our QFTs.

We now want to prove that

Theorem 3.15 (Noether). Continuous symmetries imply conserved current.

Proof. We have

$$\begin{aligned} \delta \mathcal{L}(x) &= \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi) \\ &= \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) \delta \varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi \right) \quad (\text{Leibniz}) \\ &= \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) \right] \delta \varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi \right) \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi \right) \quad (\text{by EL}), \end{aligned}$$

hence

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} X(\varphi) \right). \quad (9)$$

Define now

$$j^\mu(x) := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} X(\varphi(x)) - F^\mu(\varphi(x)).$$

Comparing Equations 6 and 9, we get

$$\partial_\mu j^\mu = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} X(\varphi) \right) - \partial_\mu F^\mu(\varphi(x)) = 0.$$

□

The proof tells us that once we know $X(\varphi)$, i.e. how the scalar field transforms, and $F(\varphi)$, i.e. the variation of the Lagrangian, we can compute the conserved current.

Lecture 5

Let us apply the previous to a scalar field. Recall we had

$$\begin{aligned} \delta\varphi(x) &= X(\varphi(x), \partial\varphi(x)) \\ \delta\mathcal{L}(x) &= \partial_\mu F^\mu(\varphi(x), \partial\varphi(x)). \end{aligned}$$

Then

$$j^\mu(x) := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} X - F^\mu,$$

obeys $\partial_\mu j^\mu = 0$. We have four symmetries at play here. For translations, given some $\varepsilon \sim c$,

$$\begin{aligned} x^\mu &\rightarrow x^\mu \varepsilon^\mu, \\ \varphi(x) &\rightarrow \varphi(x + \varepsilon) \\ &\simeq \varphi(x) + \varepsilon^\mu \partial_\mu \varphi(x) + O(\varepsilon^2), \end{aligned}$$

up-to first order. Hence

$$\begin{aligned} \delta\varphi(x) &= +\varepsilon^\mu \partial_\mu \varphi(x) \\ \delta\mathcal{L}(x) &= +\varepsilon^\mu \partial_\mu \mathcal{L}(x) \\ \delta\varphi(x) &= \varepsilon^\nu X_\nu(\varphi) \\ \delta\mathcal{L}(x) &= \varepsilon^\nu \partial_\mu F_\nu^\mu(\varphi), \end{aligned}$$

thus this gets us

$$\begin{aligned} X_\nu &= \partial_\nu \varphi \\ F_\nu^\mu &= \delta_\nu^\mu \mathcal{L}. \end{aligned}$$

By Noether's Theorem, we get four conserved currents, which we encode in the following tensor:

Definition 3.16 (Energy-Momentum Tensor). The **energy-momentum tensor** is defined as

$$T_\nu^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \partial_\nu \varphi - \delta_\nu^\mu \mathcal{L}. \quad (10)$$

We can show this is conserved by computing for each $\nu \in \{0, \dots, 3\}$ we get

$$\partial_\mu T_\nu^\mu = 0.$$

This gives, for $\nu = 0$,

$$E = \int_{\mathbb{R}^3} d^3x T^{00},$$

and for $\nu = i \in \{1, 2, 3\}$

$$\begin{aligned} (\vec{\varphi})^i &= \int_{\mathbb{R}^3} d^3x T^{0i} \\ p^\nu &= \int_{\mathbb{R}^3} d^3x T^{0\nu}. \end{aligned}$$

Example 3.17. Let us consider the case of the free scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2,$$

hence

$$\frac{\partial \mathcal{L}}{\partial_\mu \varphi} = \partial^\mu \varphi,$$

which by Equation 10 becomes

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}.$$

By setting $\mu = \nu = 0$, we extract

$$\begin{aligned} E &= \int_{\mathbb{R}^3} d^3x T^{00} \\ &= \int_{\mathbb{R}^3} d^3x \left(\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} |\vec{\nabla} \varphi|^2 + \frac{1}{2} m^2 \varphi^2 \right) \geq 0, \end{aligned}$$

where $\dot{\varphi} = \partial_t \varphi$.

Lecture 6

4 Canonical Quantization

We'll now start the process of quantisation, the process to construct a quantum field theory out of a given field theory.

This will involve the process of canonical quantisation.

Our recipe for constructing a quantum field theory out of a classical one will be closely based on the recipe one follows to construct quantum mechanics of some finite dimensional mechanical system, given its classical mechanics in the form of a classical Lagrangian or Hamiltonian.

But in order to do this, we need to formulate classical dynamics in the right way. Though the Lagrangian formalism has some advantages when it, as discussed in the previous lectures, the Hamiltonian formulation is more useful, especially when it comes

to quantization. We will cover in AQFT a direct approach to quantisation that doesn't need Hamiltonians, through path integrals.

We start with a system with N degrees of freedom and consider a Lagrangian $L(\{q_i(t)\}, \{\dot{q}_i(t)\})$. Then the Hamiltonian is obtained by Legendre transform,

Definition 4.1 (Conjugate Momenta). The **conjugate momentum** is defined by

$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i(t)}. \quad (11)$$

Definition 4.2. The **Legendre transform** to define the **Hamiltonian** by

$$H(\{q_i\}, \{p_i\}) := \sum_{i=1}^N p_i \dot{q}_i - L, \quad (12)$$

eliminating the $\dot{q}_i$ by Equation 11.

Definition 4.3 (Poisson bracket). The **Poisson bracket** is an anti-symmetric pairing of functions $F = F(p, q), G = G(p, q)$ defined by

$$\{F, G\} := \sum_{i=1}^N \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i}.$$

Example 4.4. Trivial consequence of q_i, p_i being distinct degrees of freedom is that $\{q_i, q_j\} = \{p_i, p_j\} = 0$, and $\{q_i, p_j\} = \delta_{ij}$.

We use this formalism by specifying the **state** at time $t = 0$, by providing $q_i(0), p_i(0)$. These are really coordinates on a symplectic manifold called the **phase space**. Then the time evolution are generated by $H(q, p)$ by the following law for any $F(q, p)$:

$$\dot{F} = \{F, H\}. \quad (13)$$

Remark. Here we start seeing the shadow of quantum mechanics! There is a resemblance between this and Schrödinger's equation.

Example 4.5. For example,

$$\begin{aligned} \dot{q}_i &= \{q_i, H\} = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= \{p_i, H\} = -\frac{\partial H}{\partial q_i} \end{aligned}$$

These are called **Hamilton's Equations**, and are entirely equivalent to the Euler-Lagrange Equations.

Example 4.6. If we have a conserved quantity, $\dot{Q} = 0$, then this holds iff $\{Q, H\} = 0$.

The main advantage of the Hamiltonian formalism when restricted to classical dynamics is the ability to eliminate time derivatives, allowing us to just think of the geometry of the phase space as a manifold. However, for our purposes, this is the starting point of the process of quantisation.

4.1 Quantisation

In quantum mechanics, states are vectors (really rays, equiv. classes of) $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. The observables $F(q, p)$ replaced by Hermitian operators $\hat{F} : \mathcal{H} \rightarrow \mathcal{H}$.

The simplest form of quantisation is called "*Canonical Quantisation*", and was discovered by Dirac. It essentially amounts to swapping the Poisson bracket $\{-, -\}$ with a multiple of a commutator $\frac{1}{i\hbar}[-, -]$.

Now $q_i \rightarrow \hat{q}_i, p_i \rightarrow \hat{p}_i$, and, though $[\hat{q}_i, \hat{q}_i] = [\hat{p}_i, \hat{p}_i] = 0$, we have the non-trivial commutator

$$[\hat{q}_i, \hat{p}_i] = \frac{1}{i\hbar} \delta_{ij}.$$

Following this rule, we have to replace the classical Hamiltonian $H(p, q)$ with an operator $\hat{H}(\hat{p}, \hat{q})$. But if we have a non-trivial combination of ps and qs , such as $p^2 q^2$, then it is underspecified which combination of them we should pick when they stop commuting as operators.

Now, Schrödinger's equation is given as

$$+i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle,$$

by Ehrenfest's Theorem.

4.2 Towards Quantising Field Theories

Field theories can be thought as an infinite dimensional dynamical system. The $q_i(t)$ are replaced with $\varphi(\vec{x}, t)$.

In applying the Hamiltonian formalism, we need to work with a particular notion of time, so it forces us to abandon overt Lorentz invariance. So we need to check after the fact that our choices of inertial frame is really Lorentz invariant.

Our main obstacle now is that the i label, discrete, is replaced by a continuous $\vec{x} \in \mathbb{R}^3$. Many issues in field theory stem from this.

There are two separate types of "*infinities*" that show up here:

- (i) IR, or large distance problems, a problem due to $\mathbb{R}^n$ not being compact, and $\vec{x}$ not being bounded.
- (ii) UV, continuous infinity in $\vec{x}$, which gives us problems in the very small.

To deal with both of these, we need to do something drastic. To deal with (i) we put the system in a box, and impose boundary conditions. To deal with (ii), we discretise space, and work with a **space lattice**.

Definition 4.7. The **conjugate momentum** for the field variable φ is defined by

$$\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x}, t)}.$$

Now the Legendre transform becomes

$$\mathcal{H}(\vec{x}, t) := \pi(\vec{x}, t) \dot{\varphi}(\vec{x}, t) - \mathcal{L}.$$

Eliminating the time derivatives we get, $\mathcal{H} = \mathcal{H}(\varphi(\vec{x}, t), \pi(\vec{x}, t))$ the **Hamiltonian density**.

To actually get the Hamiltonian, we integrate

$$H = \int_{\mathbb{R}^3} d^3x \mathcal{H}(\vec{x}, t).$$

Example 4.8. Consider our generic scalar field theory in 4-dimensions

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi) \\ &= \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}|\vec{\nabla}\varphi|^2 - V(\varphi).\end{aligned}$$

Now compute the conjugate momentum

$$\pi(\vec{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}(\vec{x}, t)} = \dot{\varphi}(\vec{x}, t).$$

Thus this gives

$$\begin{aligned}\mathcal{H} &= \pi\dot{\varphi} - \mathcal{L} \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi) \\ &= T^{0,0},\end{aligned}$$

the 0-0-component of the energy-momentum tensor defined in the last lecture. Hence this gets us

$$H = \int_{\mathbb{R}^3} d^3x \frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi) = E,$$

another point of contact between our scalar field theory and our classical formalism. We can now apply canonical quantisation.

Start with $\{q_i, p_j\} = \varphi_{ij}$, and turn it into $\{\varphi(\vec{x}), \pi(\vec{y}, t)\} = \delta^{(3)}(\vec{x} - \vec{y})$, the Dirac- δ , by our general framework by which sums in contractions get replaced by integrals.

We now suppress t , and replace $\varphi(\vec{x}) \rightarrow \hat{\varphi}(\vec{x})$, and $\pi(\vec{x}) \rightarrow \hat{\pi}(\vec{x})$. We have the commutator relations

$$[\hat{\varphi}(\vec{x}), \hat{\varphi}(\vec{y})] = 0 \tag{14}$$

$$[\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0 \tag{15}$$

$$[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i\hbar\delta^{(3)}(\vec{x} - \vec{y}). \tag{16}$$

The Hamiltonian operator is now

$$\hat{H} = \int_{\mathbb{R}^3} d^3x \frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}|\vec{\nabla}\hat{\varphi}(\vec{x})|^2 + V(\hat{\varphi}).$$

In general, making sense of the above for an interacting field theory is a very hard problem. If we follow the naive answer, it's really hard to get finite eigenvalues. In physics, we care about interactive particles, depending on some coupling-constant. But we can only deal with non-interactive particles, in free field theories. We will see in this lecture that we can quantise free field theories, in the sense that we can work out the Hamiltonian, and compute its spectra.

However, in the real world, we're often close to free field theory, where the problem is tractable, and we can use perturbative methods.

Definition 4.9 (Free Field Theory). We say a field theory is **free** if the potential

in the Lagrangian density is quadratic

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2.$$

This has as a consequence that the equations of motions are linear, i.e. the Klein-Gordon equations for massive field theory which is a linear equation,

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0. \quad (17)$$

By linearity, we can solve this by a Fourier Transform:

$$\varphi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \tilde{\varphi}(\vec{p}, t).$$

An in Fourier space, the equations of motion become differential equations in time-only, i.e. it gets rid of the derivatives in favour of powers of momenta:

$$\left(\frac{\partial^2}{\partial t^2} + |\vec{x}|^2 + m^2 \right) \hat{\varphi}(\vec{p}, t) = 0.$$

The solutions are given by linear combinations of various frequency waves

$$\hat{\varphi}(\vec{p}, t) = A_{\vec{p}} e^{i\omega_{\vec{p}} t} + B_{\vec{p}} e^{-i\omega_{\vec{p}} t},$$

where $\omega_{\vec{p}}^2 = |\vec{p}|^2 + m^2$, with some conditions like $A_{\vec{p}}^* = B_{-\vec{p}}$ necessary if we want to keep our solution real-valued.

This solution will allow us to write out the free field theory as an infinite set of decoupled (complex) simple harmonic oscillators.

Lecture 7

Let us now go back to the action for the free field theory

$$\begin{aligned} S &= \int dt \int d^3 x \mathcal{L}(\vec{x}, t) \\ &= \frac{1}{2} \int dt \int \frac{d^3 p}{(2\pi)^3} \tilde{\varphi}^*(\vec{p}, t) \left(-\frac{\partial^2}{\partial t^2} - |\vec{p}|^2 - m^2 \right) \tilde{\varphi}(\vec{p}, t). \end{aligned}$$

This gives an infinite set, labelled by $\vec{p} \in \mathbb{R}^3$ of complex **decoupled** simple harmonic oscillators. When we compute the spectra, we will see this corresponds to an infinite set of non-interactive particles.

4.3 Quantising Simple Harmonic Oscillators

To quantise this, we can quantise each simple harmonic oscillators (SHO), then add it back up. Each (quantum) SHO is characterised by its angular frequency ω . The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \dot{q}^2 - \frac{w^2}{2} q^2 \\ \implies H &= \frac{p^2}{2} - \frac{w^2}{2} q^2. \end{aligned}$$

Quantising this, we promote $q \rightarrow \hat{q}, p \rightarrow \hat{p}$, respecting the relations

$$\begin{aligned} [\hat{q}, \hat{q}] &= 0 \\ [\hat{p}, \hat{p}] &= 0 \\ [\hat{q}, \hat{p}] &= i\hbar. \end{aligned}$$

Normally, we would look for a representation of this algebra, but for this specific case, we can define creation and annihilation operators.

$$\begin{aligned}\hat{a} &:= \sqrt{\frac{w}{2}}\hat{q} + \frac{i}{\sqrt{2\omega}}\hat{p} \\ \hat{a}^\dagger &:= \sqrt{\frac{w}{2}}\hat{q} - \frac{i}{\sqrt{2\omega}}\hat{p}.\end{aligned}$$

| Warning. These are not observables, as they are not Hermitian!

By the commutation relation of $\hat{q}, \hat{p}$, we get

$$[\hat{a}, \hat{a}^\dagger] = i\hbar. \quad (18)$$

We can use this to re-express our Hamiltonian as

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}\hat{q}^2 \\ &= \frac{1}{2}\omega(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\ &= \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar).\end{aligned}$$

| Definition 4.10 (Number Operator). The **number operator** $\hat{N}$ is defined as $\hat{a}^\dagger\hat{a}$.

We have

$$\begin{aligned}[\hat{H}, \hat{a}] &= -\omega\hbar\hat{a} \\ [\hat{H}, \hat{a}^\dagger] &= +\omega\hbar\hat{a}^\dagger.\end{aligned}$$

Thus for an energy eigenstate $|E\rangle$, i.e.

$$\hat{H}|E\rangle = E|E\rangle,$$

we get by acting with our "ladder operators"

$$\begin{aligned}\hat{H}(\hat{a}^\dagger|E\rangle) &= (E + \hbar\omega)(\hat{a}^\dagger|E\rangle) \\ \hat{H}(\hat{a}|E\rangle) &= (E - \hbar\omega)(\hat{a}|E\rangle).\end{aligned}$$

To fix the quantisation, we observe that either

- (i) The spectrum is unbounded below,
- (ii) There exists $|0\rangle \in \mathcal{H}$ such that $\hat{a}|0\rangle = 0$.

Then, since option (ii) is the only physical one, we get

$$\begin{aligned}\hat{H}|0\rangle &= \left(\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega\right)|0\rangle \\ &= \frac{1}{2}\hbar\omega|0\rangle.\end{aligned}$$

Exercise. Check that $\hat{a}^\dagger$ does not annihilate any excited states.

This gets us an infinite tower of excited states.

Definition 4.11 (Excited States). We label our **excited states** for $n \in \mathbb{Z}_{\geq 0}$ by

$$|n\rangle := (\hat{a}^\dagger)^n |0\rangle.$$

By definition, we have $\hat{N}|n\rangle = n|n\rangle$. Moreover,

$$\hat{H}|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle,$$

and the energy $E_n = (n + \frac{1}{2})\hbar\omega$, which is not normalised in general, i.e. $\langle n|n\rangle \neq 1$.

4.4 Back to Field Theory

We have

$$\hat{\varphi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \times \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[\hat{a}_{\vec{p}} e^{ip \cdot x} + \hat{a}_{\vec{p}}^\dagger e^{-ip \cdot x} \right] \quad (19)$$

$$\hat{\pi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \times (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left[\hat{a}_{\vec{p}} e^{ip \cdot x} - \hat{a}_{\vec{p}}^\dagger e^{-ip \cdot x} \right]. \quad (20)$$

Claim. The commutation relations given in Equations 14 are equivalent to

$$\begin{aligned} [\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}] &= [\hat{a}_{\vec{p}}^\dagger, \hat{a}_{\vec{q}}^\dagger] = 0, \\ [\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \end{aligned}$$

Proof. We check

$$\begin{aligned} [\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] &= -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \times \\ &\quad \left[\hat{a}_{\vec{p}} e^{ip \cdot x} + \hat{a}_{\vec{p}}^\dagger e^{-ip \cdot x}, \hat{a}_{\vec{q}} e^{iq \cdot y} - \hat{a}_{\vec{q}}^\dagger e^{-iq \cdot y} \right], \end{aligned}$$

by Equations 19. By Equations 18 the only interesting commutators are the inner and the outer ones, which give

$$\begin{aligned} &= \left(-e^{ip \cdot x - iq \cdot y} [\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}^\dagger] + e^{-ip \cdot x + iq \cdot y} [\hat{a}_{\vec{p}}^\dagger, \hat{a}_{\vec{q}}] \right) \\ &= -(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \left[e^{ip \cdot x - iq \cdot y} + e^{-ip \cdot x + iq \cdot y} \right]. \end{aligned}$$

We can now do

$$\int \frac{d^3q}{(2\pi)^3} \delta^{(3)}(\vec{p} - \vec{q}) \dots,$$

which sets $\vec{p} = \vec{q}$, and thus gives us

$$\begin{aligned} [\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] &= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left[e^{ip \cdot (\vec{x} - \vec{y})} + e^{-ip \cdot (\vec{x} - \vec{y})} \right] \\ &= i\delta^{(3)}(\vec{x} - \vec{y}). \end{aligned}$$

□

We then want to evaluate the field-theoretic Hamiltonian in terms of the creation and annihilation operators, by plugging our Fourier expansions into the Hamiltonian. We do this by taking

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2} \hat{\pi}^2(\vec{x}) + \frac{1}{2} |\vec{\nabla} \hat{\varphi}(\vec{x})|^2 + m^2 \hat{\varphi}^2$$

and expand it out by writing out the squares as double integrals with a Kronecker-delta term. We then note that

- (i) All terms look like $e^{\pm i\vec{p}\cdot\vec{x}}$ or $e^{\pm i\vec{q}\cdot\vec{x}}$,
- (ii) We then do the d^3x integral, getting $(2\pi)^3 \delta^{(3)}(\pm\vec{p}\pm\vec{q})$ integral, which sets $\vec{q} = \pm\vec{p}$,
- (iii) Then look at the operator dependencies with 4-terms $\hat{a}_{\vec{p}}\hat{a}_{-\vec{p}}, \hat{a}_{\vec{p}}^\dagger\hat{a}_{-\vec{p}}^\dagger, \hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger, \hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}}$,
- (iv) We then collect coefficients, setting $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$, getting

$$\begin{aligned} \hat{H} = & \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \times \\ & (-\omega_{\vec{p}}^2 + |\vec{p}|^2 + m^2)(\hat{a}_{\vec{p}}\hat{a}_{-\vec{p}} + \hat{a}_{\vec{p}}^\dagger\hat{a}_{-\vec{p}}^\dagger) \times \\ & (\omega_{\vec{p}}^2 + |\vec{p}|^2 + m^2)(\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger + \hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}}). \end{aligned}$$

- (v) The first term simplifies to zero, the second to a factor of $2\omega_{\vec{p}}^2$, so the integral gets evaluated to

$$\hat{H} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \times [\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger + \hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}}]. \quad (21)$$

Note that this really gives an continuously-varying family of decoupled simple Harmonic oscillators, which only happens for free theories.

Lecture 8

Let us fully make the comparison between operator algebras and Hamiltonian for our Quantum Mechanics SHO and the field-theoretic one.

$$\begin{aligned} [\hat{a}_i, \hat{a}_i^\dagger] &= \hbar \delta_{ij} \\ \hat{H} &= \sum_{i=0}^n \frac{\omega_i}{2} [\hat{a}_i\hat{a}_i^\dagger + \hat{a}_i^\dagger\hat{a}_i]. \end{aligned}$$

The kind of infinity appearing in our field-theoretic Hamiltonian is not an entirely harmless one, and as we will see, it will introduce the first kind of divergence.

Let us now move to compute the spectra of our Hamiltonian in Equation 21, first by considering the **vacuum state** $|0\rangle$, which obeys $\hat{a}_{\vec{p}}|0\rangle = 0 \forall \vec{p} \in \mathbb{R}^3$, with the vacuum energy E_0 given by $\hat{H}|0\rangle = E_0|0\rangle$. Before we do this though, we need to put Equation 21 in normal ordering, by rewriting

$$\begin{aligned} \hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger + \hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}} &= 2\hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}} + (2\pi)^3 \delta^{(3)}(0) \\ \implies \hat{\pi} &= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \hat{a}_{\vec{p}}^\dagger\hat{a}_{\vec{p}} + \frac{1}{2} \int d^3p \omega_{\vec{p}} \delta^{(3)}(0), \end{aligned}$$

which if we take seriously, it would mean that E_0 is given by

$$E_0 = \frac{1}{2} \int d^3p \omega_{\vec{p}} \delta^{(3)}(\vec{0}) = \infty.$$

We observe that our expression for the vacuum energy is infinite in two different ways:

- (i) IR: large distances/low energies, the fact that $\vec{p}$ takes values in the whole of $\mathbb{R}^3$. Recall the usual representation of $\delta^{(3)}$ as an integral

$$(2\pi)^3 \delta^{(3)}(\vec{p}) = \int_{\mathbb{R}^3} d^3x e^{i\vec{p}\cdot\vec{x}}.$$

This is a tame divergence, as the cure here is just to work with a space of large but finite volume $L \gg m^{-1}$, and take $x \in [-\frac{L}{2}, \frac{L}{2}]$. This means that

$$(2\pi)^3 \delta^{(3)}(\vec{p}) = \lim_{L \rightarrow \infty} \int_{-\frac{L}{2}}^{+\frac{L}{2}} d^3x e^{i\vec{p}\cdot\vec{x}} = \lim_{L \rightarrow \infty} V_L,$$

where $V_L = L^3$.

- (ii) UV: the cure for this divergence is to move from computing vacuum energy to vacuum energy densities, i.e.

$$\varepsilon_0 := \lim_{L \rightarrow \infty} \frac{E_0^{(L)}}{V_L},$$

where

$$E_0^{(L)} := \int_{-\frac{L}{2}}^{+\frac{L}{2}} d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_{\vec{p}}$$

$$\varepsilon_0 := \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_{\vec{p}},$$

where $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$. This shows that we're really integrating over the zero-point energies of all these oscillators. But we are still dealing with arbitrarily high frequencies, coming from arbitrarily large wave numbers of the oscillator modes concerned. This is an example of a UV divergence, related with high momenta/small wavelengths, which shows up even when we put our theory in a box. To solve this, we need to think about the behaviour of the theory at short distance. There is a dirty trick to solve this, and a more conceptual one.

We fix our UV woes by introducing an UV cut-off, some energy scale above which we cut off the momentum. This means that both position and momenta are bounded by some box. For it to be a cut off, we want it to be much larger than the energy scale and mass scale of our theory, i.e. our UV cut-off $\Lambda \gg m \gg L^{-1}$. This yields

$$\begin{aligned} \varepsilon_0^{(\Lambda)} &= \int_{|\vec{p}|^2 \ll \Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_{\vec{p}} \\ &= \frac{4\pi}{2(2\pi)^3} \int_0^\Lambda |\vec{p}|^2 d|\vec{p}| \sqrt{|\vec{p}|^2 + m^2} \\ &= \frac{1}{16\pi^2} \Lambda^{-1} \left[1 + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right) \right]. \end{aligned}$$

The alternative to this momentum-box is to put the theory on a space(time) lattice. So instead of working with a manifold $\mathbb{R}^3$, we work with some lattice $\mathbb{Z}^3$. There the UV cut-off $\Lambda \propto 2\pi a^{-1}$, where a is the lattice spacing. In a lattice, the energy density scales as

$$\varepsilon_0 \propto \frac{1}{a^4} \frac{1}{16\pi^2}.$$

Though we can avoid it for free field theories, for interactive field theories we need to renormalise seriously, before taking the continuum limit $a \rightarrow 0, \Lambda \rightarrow \infty$.

In the specific case of free field theory, instead of using UV cut-offs, we can use an alternative quantisation. Remember that when quantising, we have some liberty in picking various alternatives. This is the subject of normal ordering:

Definition 4.12. Let $\hat{X}$ be a monomial in $\hat{a}, \hat{a}^\dagger$. Then the **normal ordering** : $\hat{X}$: of $\hat{X}$ is obtained by placing all quantisation operators to the right.

Example 4.13. We have

$$: \hat{a}(\hat{a}^\dagger)^2 \hat{a} \hat{a}^\dagger : = (\hat{a}^\dagger)^3 \hat{a}^2.$$

For the free field theory, this gives the new Hamiltonian

$$\begin{aligned} \hat{H}_{\text{Normal}} =: \hat{H} &:= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} : [\hat{a}_{\vec{p}} \hat{a}_{\vec{p}}^\dagger + \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}] : \\ &= \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}. \end{aligned}$$

Hence $\hat{H}_{\text{Normal}} |0\rangle = 0 \implies (\varepsilon_0)_{\text{Normal}} = 0$, as a consequence of annihilation being to the right.

Question. Do we care about additive ambiguity/additive infinities? Without coupling to gravity, only ΔE are observable.

Answer. We'll eventually have to consider gravity, the only place where the absolute energies matter. Recall that on the right of Einstein's Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 9\pi G T_{\mu\nu}.$$

In QFT, far from Plank-length, we will have

$$\langle 0 | \hat{T}_{\mu\nu} | 0 \rangle = \varepsilon_0 g_{\mu\nu}.$$

This is worrying because the thing on the right-hand side is known as the cosmological constant, which is not zero! The latest consensus in cosmology is that, give $T_{\mu\nu} \propto \lambda g_{\mu\nu}$, $\lambda \propto (10^{-3} \text{eV})^2$. We cannot get this in quantum field theory.

Lecture 9

Recall that our quantisation of free field theories yielded

$$\begin{aligned} [\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ \hat{H} &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} [\hat{a}_{\vec{p}} \hat{a}_{\vec{p}}^\dagger + \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}], \end{aligned}$$

with $\omega_{\vec{p}} = +\sqrt{|\vec{p}|^2 + m^2}$.

4.5 Excited States

Our vacuum state corresponds to putting each decoupled SHO in its ground state. We will study the excited states by acting with raising the ground state by $\hat{a}_{\vec{p}}^\dagger$.

Definition 4.14. We define for $\vec{p} \in \mathbb{R}^3$, $|\vec{p}\rangle := \hat{a}_{\vec{p}}^\dagger |0\rangle$.

We can either consider the normal-ordered Hamiltonian, or subtract off the ground energy:

$$\hat{H} - E_0 =: \hat{H} := \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}} \hat{a}_{\vec{q}}^\dagger \hat{a}_{\vec{q}},$$

in other words we want to consider

$$\begin{aligned} (\hat{H} - E_0) |\vec{p}\rangle &= (\hat{H} - E_0) \hat{a}_{\vec{p}}^\dagger |0\rangle \\ &= \hat{a}_{\vec{p}}^\dagger (\hat{H} - E_0) |0\rangle + [(\hat{H} - E_0), \hat{a}_{\vec{p}}^\dagger] |0\rangle, \end{aligned}$$

where the commutator obeys

$$\begin{aligned} [(\hat{H} - E_0), \hat{a}_{\vec{p}}^\dagger] &= [(\hat{H} - E_0), \hat{a}_{\vec{p}}^\dagger] |0\rangle \\ &= \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}} [\hat{a}_{\vec{q}}^\dagger \hat{a}_{\vec{q}}, \hat{a}_{\vec{p}}^\dagger] \\ &= \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}} \hat{a}_{\vec{q}}^\dagger [\hat{a}_{\vec{q}}, \hat{a}_{\vec{p}}^\dagger] \\ &= \int d^3q \omega_{\vec{q}} \hat{a}_{\vec{q}}^\dagger \delta^{(3)}(\vec{q} - \vec{p}) \\ &= \omega_{\vec{p}} \hat{a}_{\vec{p}}^\dagger. \end{aligned}$$

This means we have an eigenstate

$$(\hat{H} - E_0) |\vec{p}\rangle = \omega_{\vec{p}} \hat{a}_{\vec{p}}^\dagger |0\rangle = \omega_{\vec{p}} |\vec{p}\rangle.$$

Thus the energy (difference) of the 1st excited state is

$$E_1 = \omega_{\vec{p}}.$$

Let us work out the momentum. The classical momentum is given

$$p^i = \int d^3x T^{0i},$$

where by our component-formula for T we have

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}.$$

Exercise (ES2.1). Compute the canonical quantisation of the above, producing a normal ordered momentum operator.

With similar manoeuvres as the ones we applied, we get

$$\hat{\vec{p}} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}.$$

This has the consequence that

$$\hat{\vec{p}} |\vec{p}\rangle = \vec{p} |\vec{p}\rangle,$$

hence this is really a momentum operator. This shows that the state $|\vec{p}\rangle$ really does correspond to a relativistic particle of mass m , momentum $\vec{p} \in \mathbb{R}^3$, and energy $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$.

We also want to check that we have a spineless solution. This is sort of contained in the fact that we only have an excited state E_1 , and not $2l + 1$, which would correspond to a spin- l particle. But we can also check that the angular momentum $\vec{J}$ is zero in the rest frame state.

Definition 4.15. The **rest frame state** is defined as $|\vec{0}\rangle = \lim_{\vec{p} \rightarrow \vec{0}} |\vec{p}\rangle$.

Claim. We have $\hat{J}|\vec{0}\rangle = 0$.

4.6 Multi-particle States

Question. What about higher states?

Definition 4.16. We define

$$|\vec{p}_1, \dots, \vec{p}_n\rangle := \hat{a}_{\vec{p}_1}^\dagger \cdots \hat{a}_{\vec{p}_n}^\dagger |0\rangle.$$

Computing the energy $(E - E_0)$ is easy, since all these operators commute with each other, and

$$(E - E_0) = \sum_{a=1}^n E_{\vec{p}_a},$$

$$\vec{p} = \sum_{a=1}^n \vec{p}_a.$$

Note that we do not need to assume $\vec{p}_i \neq \vec{p}_j$ for $i \neq j$.

We can check bose statistics, for instance, checking that the particles we get are identical:

$$\begin{aligned} |\vec{p}_1, \vec{p}_2\rangle &= \hat{a}_{\vec{p}_1}^\dagger \hat{a}_{\vec{p}_2}^\dagger |0\rangle \\ &= \hat{a}_{\vec{p}_2}^\dagger \hat{a}_{\vec{p}_1}^\dagger |0\rangle = |\vec{p}_2, \vec{p}_1\rangle. \end{aligned}$$

Bose statistics are expected because we are dealing with an integer spin theory.

We now introduce a particle number operator

$$\hat{N} = \int \frac{d^3p}{(2\pi)^3} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}},$$

which obeys

$$\hat{N} |\vec{p}_1, \dots, \vec{p}_n\rangle = n |\vec{p}_1, \dots, \vec{p}_n\rangle.$$

For our free field theory, this operator commutes with the Hamiltonian $[\hat{H}, \hat{N}] = 0$, i.e. it represents the QFT equivalent of conserved quantity.

4.7 Choices of Normalisations

Choose for the moment $\langle 0|0\rangle = 1$. Then

$$\langle \vec{p}|\vec{q}\rangle = \langle 0|\hat{a}_{\vec{p}}\hat{a}_{\vec{q}}^\dagger|0\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}).$$

This is not ideal, as it is not relativistically invariant, since the " $\delta^{(3)}$ " bit is not four-dimensional. We can express this in terms of the corresponding the following **completeness identity/resolution of unity**:

$$\hat{1} = \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}|.$$

The left hand-side is relativistically invariant, but the measure is not. So we will tamper with the metric, and then absorb those factors within the gets.

Our Lorentz invariant measure will be given by

$$\int d\mu_p = \int d^4p \delta(p_\mu p^\mu - m^2) \Theta(p_0),$$

where

$$\Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

We can check that $p^\mu p_\mu = m^2$, is solved by p^0 and $p^\mu = (p_0, \vec{p})$ iff $p_0 = +\sqrt{|\vec{p}|^2 + m^2} = E_{\vec{p}}$. Let us simplify this further

$$\int d\mu_p = \int d^3p \int_0^\infty \frac{d(p_0)^2}{2p_0} \times \delta(p_0^2 - |\vec{p}|^2 - m^2) = \int \frac{d^3p}{2E_{\vec{p}}}.$$

Rewriting the completeness relation

$$|p\rangle := \sqrt{2E_{\vec{p}}|\vec{p}|} |\vec{p}\rangle,$$

with $p^\mu = (E_{\vec{p}}, \vec{p})$. This gives

$$\hat{1} = \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}| = \int \frac{d\mu_p}{(2\pi)^3} |p\rangle \langle p|.$$

This will be useful when we turn to computing scattering amplitudes.

Hence from now on, we will define our multi-particle states as

$$|\vec{p}_1, \dots, \vec{p}_n\rangle := \left(\prod_{a=1}^n \sqrt{2E_{\vec{p}_a}} \right) \times \hat{a}_{\vec{p}_1}^\dagger \cdots \hat{a}_{\vec{p}_n}^\dagger |0\rangle. \quad (22)$$

Lecture 10

4.8 Time-Dependence

So far we were in Schrödinger picture, where the time dependence is baked into the states. For $\hat{\varphi}(\vec{x}) = \hat{\varphi}_S(\vec{x})$ we have

$$i \frac{d}{dt} |\psi(t)\rangle_S = \hat{H} |\psi(t)\rangle,$$

which means $|\vec{p}\rangle \rightarrow |\vec{p}(t)\rangle$, as governed by the Schrödinger's equation

$$i \frac{d}{dt} |\vec{p}(t)\rangle_S =: \hat{H} : |\vec{p}(t)\rangle = E_{\vec{p}} |\vec{p}(t)\rangle_S.$$

Since the operators are time independent, we can integrate up to get an oscillating operator

$$e^{-iE_{\vec{p}}t} |\vec{p}(0)\rangle.$$

We will now include time dependence in the operators and move to Heisenberg picture. This really amounts to an alternative choice of the basis for our Hilbert space $\mathcal{H}$. To get the Heisenberg-picture states, we define our states by undoing our time-evolution

$$|\psi\rangle_H = e^{+i\hat{H}t} |\psi(t)\rangle_S.$$

At the same time, we define the Heisenberg-picture operators

$$\hat{O}_H = e^{+i\hat{H}t} \hat{O}_S e^{-i\hat{H}t}.$$

Define H -picture field operator

$$\hat{\varphi}(t, \vec{x}) = \hat{\varphi}(x) := e^{+i\hat{H}t} \hat{\varphi}(\vec{x}) e^{-i\hat{H}t},$$

which works by unitarity of $e^{\pm i\hat{H}t}$.

Exercise (ES2.2). Show directly that $\hat{\varphi}(t, \vec{x})$ obeys the Klein-Gordon Equations.

Let us consider the corresponding mode-expansion of $\hat{\varphi}(x)_H$.

$$\hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left((\hat{a}_{\vec{p}})_H e^{i\vec{p}\cdot\vec{x}} + (\hat{a}_{\vec{p}}^\dagger)_H e^{-i\vec{p}\cdot\vec{x}} \right),$$

where explicitly,

$$\begin{aligned} (\hat{a}_{\vec{p}})_H &= e^{+i\hat{H}t} \hat{a}_{\vec{p}} e^{-i\hat{H}t} \\ \implies \frac{d}{dt} (\hat{a}_{\vec{p}})_H &= i[\hat{H}, (\hat{a}_{\vec{p}})_H] = -iE_{\vec{p}} (\hat{a}_{\vec{p}})_H. \end{aligned}$$

This is a trivial commutator to evaluate, as for SHO we already know how ladder operators commute with $\hat{H}$. We now integrate up the differential equation:

$$\begin{aligned} (\hat{a}_{\vec{p}})_H(t) &= e^{-iE_{\vec{p}}t} (\hat{a}_{\vec{p}})_H(0) = e^{-iE_{\vec{p}}t} \hat{a}_{\vec{p}} \\ (\hat{a}_{\vec{p}}^\dagger)_H(t) &= e^{+iE_{\vec{p}}t} (\hat{a}_{\vec{p}}^\dagger)_H(0) = e^{+iE_{\vec{p}}t} \hat{a}_{\vec{p}}^\dagger. \end{aligned}$$

Hence

$$\hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left((\hat{a}_{\vec{p}}) e^{-ip\cdot x} + (\hat{a}_{\vec{p}}^\dagger)_H e^{+ip\cdot x} \right), \quad (23)$$

where $p \cdot x = p_\mu x^\mu = E_{\vec{p}}t - \vec{p} \cdot \vec{x}$. This is another way of seeing that Klein-Gordon holds.

5 Interacting QFT

Starting point for our discussion is more general Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \tilde{\varphi} \partial^\mu \tilde{\varphi} - V(\tilde{\varphi}).$$

The Hamiltonian

$$H = \int d^3x \left(\frac{1}{2} \dot{\tilde{\varphi}}^2 + \frac{1}{2} |\vec{\nabla} \tilde{\varphi}|^2 + V(\tilde{\varphi}) \right).$$

2022-10-27 To attain the minimum value, we set $\tilde{\varphi}(\vec{x}, t) \equiv \tilde{\varphi}_0$, the (global) minimum of potential.
12:25:54 Hence

$$\frac{\partial V}{\partial \tilde{\varphi}} \Big|_{\tilde{\varphi}=\tilde{\varphi}_0} = 0, \quad (24)$$

$$\frac{\partial^2 V}{\partial \tilde{\varphi}^2} \Big|_{\tilde{\varphi}=\tilde{\varphi}_0} \geq 0. \quad (25)$$

Without loss of generality, we set $V(\tilde{\varphi}_0) = 0$. We now consider an expansion of the field around this minimum, by replacing it with its Taylor series. By change of variable and assuming that V is analytic, we define

$$\begin{aligned} \varphi(x) &:= \tilde{\varphi}(x) - \tilde{\varphi}_0, \\ V(\tilde{\varphi}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n V}{\partial \tilde{\varphi}^n} \right) \Big|_{\tilde{\varphi}=\tilde{\varphi}_0} \cdot \varphi^n. \end{aligned}$$

We want to understand $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, free and interactive part decomposition. We use our understanding of the free massive QFT to write

$$\mathcal{L}_0 := \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2,$$

where

$$m^2 := \frac{\partial^2 V}{\partial \tilde{\varphi}^2} \Big|_{\tilde{\varphi}=\tilde{\varphi}_0} \geq 0.$$

As for $\mathcal{L}_{\text{int}}$, we define it as

$$\mathcal{L}_{\text{int}} = - \sum_{n=3}^{\infty} \frac{\lambda_n}{n!} \varphi^n,$$

where

$$\lambda_n := \frac{\partial^n V}{\partial \tilde{\varphi}^n} \Big|_{\tilde{\varphi}=\tilde{\varphi}_0}.$$

Definition 5.1 (Couplings). We call the λ_n the **couplings** of the theory.

We can calculate only when the couplings are "small", whatever that means.

5.1 Dimensional Analysis of Couplings

The λ_n s are not in general dimensionless, so we always need to talk about small w.r.t. some energy scale. We note that $[S] = 0$ in our natural units, and moreover $[x] = -1$, and thus $[\partial_m u] = +1$, which by FT corresponds to a momentum. But then

$$\begin{aligned} S &= \int_{\mathbb{R}^{3,1}} d^4x \mathcal{L} \implies [\mathcal{L}] = +4 \\ \implies [\partial_\mu \varphi \partial^\mu \varphi] &= +4 \implies [\varphi] = +1 \\ \implies \left[\frac{\lambda_n \varphi^n}{n!} \right] &= 4 \implies [\lambda_n] = 4 - n. \end{aligned}$$

Warning. This is all highly dependent on the underlying dimension of our field theory. In our case, we only concern ourselves with dimension 3+1 field theory.

The $n = 4$ is very special, as λ_4 is the only dimensionless coupling constant. Let $E \gg m$, the **effective dimensionless parameter**

$$\widetilde{\lambda}_n = \lambda_n(E)^{n-4} \implies [\widetilde{\lambda}_n] = 0.$$

Hence our perturbation will eventually breakdown outside low-energy scales. Three cases follow

- (i) For $n < 4$, we have **relevant coupling**, involving negative energy scaling. Here we have a weak coupling at high energy. Perturbation theory will only be good at $\lambda_n \ll (E)^{4-n}$, hence $\widetilde{\lambda}_n \ll 1$. For example, if we choose $\lambda_n \ll (m)^{4-n}$, the coupling will be small at all energies and perturbation theory will be good.
- (ii) For $n = 4$, we have **marginal coupling**, perturbation theory will be good for $\lambda_4 \ll 1$.
- (iii) For $n > 4$, we have **irrelevant couplings**, and $\widetilde{\lambda}_n = \lambda_n(E)^{+\text{positive}}$. Here perturbation theory will be good at low energies only. When we look at loop this will get into non-renormalisable quantum field theories.

Example 5.2 (φ^4). An example of a genuinely renormalisable theory is the φ^4 -theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4,$$

which has marginal coupling.

Example 5.3 (φ^3). We can also do this for the φ^3 -theory, but that has an unbounded potential, so we really want to consider a $\varphi^3 - \varphi^4$ theory.

Example 5.4 (Non φ^6). The φ^6 -theory is not renormalisable.

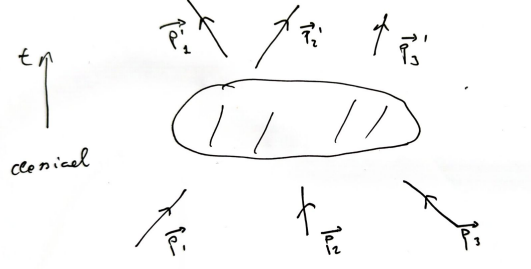


Figure 1: A classical scattering experiment.

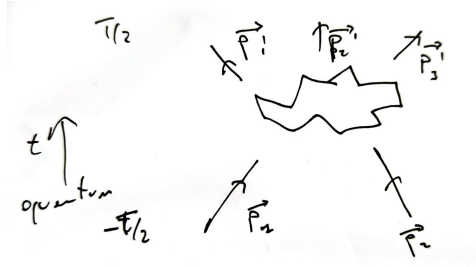


Figure 2: A QFT scattering experiment.

5.2 Scattering

The basic experiment in particle physics we prepare beams of particles and then either aim them to other beams or against some target, and then observe the results of these scattering experiments. We will now set up the formalism for scattering in QFT, starting from the picture in classical field theory.

The basic setup is given by Figures 1 and 2 of short-range acting particles. In the classical picture, depicted in Figure 1, the same particle emerge from the collision, and moreover, we have conservation of total energy and momenta. Beyond this, what happens depends on the interaction. Under the assumption that our interaction is by short-range forces, we get particles moving freely for $t \rightarrow \pm\infty$.

Scattering in the quantum theory behaves differently, as shown in Figure 2. We may choose to localise our wave-packets. Particle number will not be conserved, but momentum and energy will, as there are associated Noether charges. The laws of QM ask us to compute an amplitude

$$\mathcal{A}_{i \rightarrow f} = \lim_{T \rightarrow \infty} \langle f, t = +T/2 | e^{-i\hat{H}t} | i, t = -T/2 \rangle_S,$$

for some initial $|i, t = -T/2\rangle_S$, evolved with time-evolution. The obvious place to start is the φ^4 -theory, which when we look at loop-level expansions we'll see is the highest renormalisable power, something not visible at tree-level.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \\ \Rightarrow \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{\text{int}} = \frac{1}{2} \int_{\mathbb{R}^3} d^3x \left(\hat{\pi}^2 + |\vec{\nabla} \hat{\varphi}|^2 + m\hat{\varphi}^2 \right) + \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \hat{\varphi}^4(\vec{x}). \end{aligned}$$

In particular, we will see that $\hat{H}_0$ and $\hat{H}_{\text{int}}$ will not commute, and moreover $\hat{N}$ is no longer a symmetry. To deal with this, we will set $\lambda \ll 1$ and use a perturbation-theory expansion.

A key assumption that will come back to will be that particles are free at $t \rightarrow \pm\infty$, by locality. This has the consequence that $|i\rangle, |f\rangle$ are eigenstates of $\hat{H}_0$. So initial and final states have definite number of particles with a definite set of momenta.

Let us consider which picture we want to study this in, i.e. which basis for our Hilbert space. We will work in the **interaction picture**, which is the Heisenberg picture w.r.t. $\hat{H}_0$ and Schrödinger w.r.t. $\hat{H}_{\text{int}}$. In this picture, the states are given by

$$|\psi(t)\rangle_{\mathcal{I}} := e^{+i\hat{H}_0 t} |\psi(t)\rangle_S \quad (26)$$

$$\hat{O}_{\mathcal{I}}(t) := e^{+i\hat{H}_0 t} \hat{O}_S e^{-i\hat{H}_0 t}. \quad (27)$$

Let us consider the time-evolution, by considering the Schrödinger Equation

$$\begin{aligned} i \frac{d}{dt} |\psi(t)\rangle_S &= \hat{H} |\psi(t)\rangle_S \\ |\psi(t)\rangle_S &= e^{-i\hat{H}t} |\psi(0)\rangle_S. \end{aligned}$$

In the interaction picture,

$$\begin{aligned} i \frac{d}{dt} |\psi(t)\rangle_{\mathcal{I}} &= i \frac{d}{dt} \left(e^{+i\hat{H}_0 t} |\psi(t)\rangle_S \right) \\ &= -\hat{H}_0 e^{+i\hat{H}_0 t} |\psi(t)\rangle_S + e^{+i\hat{H}_0 t} i \frac{d}{dt} |\psi(t)\rangle_S \\ &= e^{+i\hat{H}_0 t} (\hat{H}_0 + \hat{H}_{\text{int}}) |\psi(t)\rangle_S \\ &= e^{+i\hat{H}_0 t} \hat{H}_{\text{int}} e^{-i\hat{H}_0 t} |\psi(t)\rangle_S. \end{aligned}$$

This gives

$$i \frac{d}{dt} |\psi(t)\rangle_{\mathcal{I}} = \hat{H}_{\mathcal{I}}(t) |\psi(t)\rangle_{\mathcal{I}} \quad (28)$$

$$\hat{H}_{\mathcal{I}}(t) := e^{+i\hat{H}_0 t} \hat{H}_{\text{int}} e^{-i\hat{H}_0 t}. \quad (29)$$

Warning. A time-dependent Hamiltonian is a genuine feature of interaction-picture, as there are non-vanishing commutators there.

Question. Can we solve Equation 28?

Let us get some intuition from ODEs, removing for a moment the operators, as

$$\begin{aligned} i \frac{d}{dt} F(t) &= g(t) F(t) \\ \implies F(t) &= \exp \left(\frac{1}{i} \int_0^t dt' g(t') \right) \times F(0), \end{aligned}$$

by using an integrated factor. The operator case is much harder, as in general we have $[H_I(t), H_I(t')] \neq 0$. The answer will be through the concept of **time-ordering**.

Definition 5.5 (Time-Ordering). Let $\hat{O}(t)$ be an operator-valued function. The **time-ordering** is defined as

$$T[\hat{O}(t_1)\hat{O}(t_2)] := \begin{cases} \hat{O}(t_1)\hat{O}(t_2) & \text{if } t_1 \geq t_2 \\ \hat{O}(t_2)\hat{O}(t_1) & \text{if } t_1 \leq t_2 \end{cases}$$

Theorem (Dyson). Equation 28 is solved by

$$|\psi(t)\rangle_{\mathcal{I}} = T \left[\exp \left(\frac{1}{i} \int_0^t dt' \hat{H}_I(t') \right) \right] |\psi(0)\rangle_S.$$

Proof. Let us expand the exponential

$$\begin{aligned} T \left[\exp \left(\frac{1}{i} \int_0^t dt' \hat{H}_I(t') \right) \right] &= T \left[1 + \frac{1}{i} \int_0^t dt' \hat{H}_I(t') + \frac{1}{2} \frac{1}{i^2} \int_0^t \int_0^t dt' dt'' \hat{H}_I(t') \hat{H}_I(t'') + \dots \right] \\ &= 1 + \frac{1}{i} \int_0^t dt' \hat{H}_I(t') + \frac{1}{i^2} \int_0^t \int_0^{t'} dt' dt'' \hat{H}_I(t') \hat{H}_I(t''). \end{aligned}$$

To check the LHS of Equation 28, we get

$$\begin{aligned} \dots &= i \frac{d}{dt} \left[T \left[\exp \left(\frac{1}{i} \int_0^t dt' \hat{H}_I(t') \right) \right] \right] \\ &= \left(\hat{H}_I(t) + \hat{H}_I(t) \frac{1}{i} \int_0^t dt' \hat{H}_I(t') + \dots \right) |\psi(0)\rangle_S \\ &= \hat{H}_I(t) \left(1 + \frac{1}{i} \int_0^t dt' \hat{H}_I(t') + \dots \right) |\psi(0)\rangle_S \\ &= \hat{H}_I(t) T \left[\exp \left(\frac{1}{i} \int_0^t dt' \hat{H}_I(t') \right) \right] |\psi(0)\rangle_S. \end{aligned}$$

□

Exercise. Check this to higher orders.

Remark. There is a very slick proof in [Ton06] that works for all orders.

Lecture 12

Let us return to our scattering experiment Figure 2. We want to compute the amplitude

$$\mathcal{A}_{i \rightarrow f} = \lim_{T \rightarrow \infty} \langle f, t = +T/2 | e^{-i\hat{H}t} | i, t = -T/2 \rangle_S = \lim_{T \rightarrow \infty} [\mathcal{M}(T)].$$

We will now work on the matrix elements of $\mathcal{M}(T)$. We have

$$\begin{aligned} M(T) &= S \left\langle f, t = +\frac{T}{2} \middle| i, t = +\frac{T}{2} \right\rangle_S \\ &= \mathcal{I} \left\langle f, t = +\frac{T}{2} \middle| i, t = +\frac{T}{2} \right\rangle_{\mathcal{I}} \\ &= \mathcal{I} \left\langle f, t = +\frac{T}{2} \middle| T \left[\exp \left(\frac{1}{i} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \hat{H}_I(t) dt \right) \right] \middle| i, t = -\frac{T}{2} \right\rangle_{\mathcal{I}}. \end{aligned}$$

Remark. This seems hard to calculate but is in a nice form to do a perturbative expansion on it.

5.3 φ^4 -Scattering

We will specialise this to the φ^4 theory.

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} = \frac{1}{2} \int_{\mathbb{R}^3} d^3x \left(\hat{\pi}^2 + |\vec{\nabla} \hat{\varphi}|^2 + m \hat{\varphi}^2 \right) + \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \hat{\varphi}^4(\vec{x}).$$

Converting to the interaction picture, we get

$$\hat{H}_I(t) = e^{+i\hat{H}_0 t} \hat{H}_{\text{int}} e^{-i\hat{H}_0 t}.$$

In particular, the Heisenberg-picture field operator is given in terms of the Schrödinger-picture field operator

$$\hat{\varphi}(x) = e^{+i\hat{H}_0 t} \hat{\varphi}(\vec{x}) e^{-i\hat{H}_0 t},$$

which gives

$$\hat{H}_I(t) = \frac{\lambda}{4!} \int_{\mathbb{R}^3} d^3x \hat{\varphi}^4(x).$$

We now split $\hat{\varphi}(x)$ in terms involving the annihilation and creation operators:

Definition 5.6. We define

$$\begin{aligned} \hat{\varphi}_+(x) &:= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \hat{a}_{\vec{p}} e^{-ip \cdot x} \\ \hat{\varphi}_-(x) &:= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \hat{a}_{\vec{p}}^\dagger e^{+ip \cdot x}, \end{aligned}$$

where again $p^\mu = (E_{\vec{p}}, \vec{p})$.

Proposition 5.7. We have $\hat{\varphi}(x) = \hat{\varphi}_+(x) + \hat{\varphi}_-(x)$.

Turing to scattering, we now choose an initial (Schrödinger-picture)

$$\begin{aligned} |i\rangle_S &= |\vec{p}_1, \dots, \vec{p}_n\rangle := \left(\prod_{a=1}^n \sqrt{2E_{\vec{p}_a}} \hat{a}_{\vec{p}_a}^\dagger \right) |0\rangle \\ |f\rangle_S &= |\vec{p}'_1, \dots, \vec{p}'_n\rangle := \left(\prod_{a=1}^n \sqrt{2E_{\vec{p}'_a}} \hat{a}_{\vec{p}'_a}^\dagger \right) |0\rangle. \end{aligned}$$

In interaction picture, we pick-up phases which we ignore

$$\begin{aligned} |i\rangle_{\mathcal{I}} &= e^{+i\hat{H}_0 \frac{T}{2}} |i\rangle_S = e^{-iE_i \frac{T}{2}} |i\rangle_S \\ |f\rangle_{\mathcal{I}} &= e^{+iE_f \frac{T}{2}} |f\rangle_S. \end{aligned}$$

Hence

$$\mathcal{A}_{i \rightarrow f} = \langle f | T \left[\exp \left(\frac{1}{i} \frac{\lambda}{4!} \int_{\mathbb{R}^{3,1}} d^4x \hat{\varphi}^4(x) \right) \right] | i \rangle.$$

5.4 Perturbation Theory

Note that this expression is really just a power series which defines the exponential, so we can just expand for $\lambda \ll 1$, and never solve exactly.

$$\begin{aligned}\mathcal{A}_{i \rightarrow f} &= \cdots = \sum_{l=0}^{\infty} \lambda^l \mathcal{A}_{i \rightarrow f}^{(l)} \\ \mathcal{A}_{i \rightarrow f}^{(l)} &= \frac{1}{l!} \left(\frac{1}{i(4!)} \right)^l \int d^4 x_1 \cdots \int d^4 x_l \langle f | T [\hat{\varphi}^4(x_1) \cdots \hat{\varphi}^4(x_l)] | i \rangle.\end{aligned}$$

We need to develop good technology to compute expectation values of time-orderings of such strings of operators.

Let us remark on the tie with normal ordering. The normal ordering of that expression is much easier to compute, so we could rephrase the problem in terms of finding the relation between the time-ordering and the normal ordering of these operators. We will do this in the next Section, which will lead up to Wick's Theorem. We will introduce vertices and links, and this will lead to the formalism of Feynman diagrams.

We will start with the problem of evaluating $T_2 := T[\hat{\varphi}(x)\hat{\varphi}(y)]$. Recall that if $x^0 > y^0$, we have $T_2 = \hat{\varphi}(x)\hat{\varphi}(y)$. We want to relate this to the creation and annihilation expression, which we can do by the $\pm$ -decomposition of $\hat{\varphi}$.

$$\begin{aligned}T_2 &= \hat{\varphi}(x)\hat{\varphi}(y) = (\hat{\varphi}_+(x) + \hat{\varphi}_-(x))(\hat{\varphi}_+(y) + \hat{\varphi}_-(y)) \\ &= \hat{\varphi}_+(x)\hat{\varphi}_+(y) + \hat{\varphi}_-(x)\hat{\varphi}_+(y) + \hat{\varphi}_-(y)\hat{\varphi}_+(x) + [\hat{\varphi}_+(x), \hat{\varphi}_-(y)] + \hat{\varphi}_-(x)\hat{\varphi}_- \\ \implies T[\hat{\varphi}(x)\hat{\varphi}(y)] &=: \hat{\varphi}(x)\hat{\varphi}(y) : + D(x-y),\end{aligned}$$

where

$$\begin{aligned}D(x-y) &= [\hat{\varphi}_+(x), \hat{\varphi}_-(y)] = \int \frac{d^3 p d^3 q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \exp(+i(q \cdot y - p \cdot x)) [\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}^\dagger] \\ &= \int \frac{d^3 p d^3 q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \exp(+i(q \cdot y - p \cdot x)) (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \exp(-ip \cdot (x - y)),\end{aligned}$$

and thus in both cases we get

$$T[\hat{\varphi}(x)\hat{\varphi}(y)] =: \hat{\varphi}(x)\hat{\varphi}(y) : + \Delta_F(x-y),$$

where Δ_F is the **Feynman propagator**

$$\Delta_F(x-y) = \begin{cases} D(x-y) & \text{if } x^0 \geq y^0 \\ D(y-x) & \text{if } x^0 \leq y^0 \end{cases}$$

We can define it in a neater way by considering its vacuum expectation value

$$\Delta_F(x-y) = \langle 0 | T [\hat{\varphi}(x)\hat{\varphi}(y)] | 0 \rangle.$$

Since the Feynman propagator is rather ubiquitous, and will correspond to the lines in the Feynman diagrams, we want to develop yet another representation for it, in integral form.

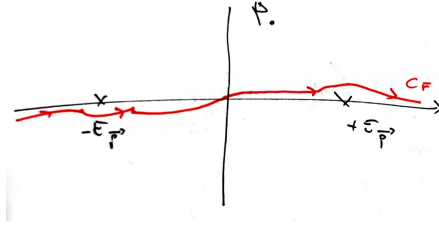


Figure 3: The Feynman contour.

5.5 Integral Representation for Δ_F

Ultimately we will try to move our perturbative expansion into Fourier space, so let us define the Fourier transformed

$$\begin{aligned} I_F(x-y) &:= \int_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip \cdot (x-y)} \\ &= \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} i e^{+i\vec{p} \cdot (\vec{x} - \vec{y})} \times \int_{C_F} dp_0 J(p_0), \end{aligned}$$

where $p^\mu = (p^0, \vec{p})$, and $p \cdot (x-y) = p_\mu (x-y)^\mu$, where the measure has been divided into a time and spatial components. Note that we are using that in the time component, we have $p_0 \equiv p^0$.

We have

$$J(p_0) := \frac{e^{-ip_0 \cdot (x-y)^0}}{(p_0)^2 - |\vec{p}|^2 - m^2}.$$

Now the integrand has poles on the real axis given by partial fractions

$$\frac{1}{p_0^2 - E_{\vec{p}}^2} = \frac{1}{2E_{\vec{p}}} \left[\frac{1}{p_0 - E_{\vec{p}}} - \frac{1}{p_0 + E_{\vec{p}}} \right],$$

at $p_0 = \pm E_{\vec{p}}$.

We can now see that

$$\text{Res}_{p^0 = \pm E_{\vec{p}}} [J(p_0)] = \pm \frac{1}{2E_{\vec{p}}} e^{\mp iE_{\vec{p}}(x^0 - y^0)},$$

where the Feynman contour given by Figure 3.

Lecture 13

Claim. We have $I_F(x-y) = \Delta_F(x-y)$.

Proof. Noting that $p^0 = \text{Re}[p^0] + i\partial m[p^0]$, we can see that $\partial(p_0) \sim e^{+\partial m(p_0) \cdot (x^0 - y^0)}$, which decays rapidly as $p^0 \rightarrow \infty$ in the lower half-plane, i.e. for $x^0 > y^0$, under the assumption that $\partial m p_0 < 0$, and in the upper half-plane for $x^0 < y^0$.

For $x^0 > y^0$, we can close the contour in the lower half-plane at infinity, see Figure 4. By the fast decay, we can use Jordan's Lemma, which ensures the contribution of the closing bit of the contour will be zero. This allows us to compute the integral by Cauchy's Theorem, i.e. just by computing the residue at the pole on $\mathbb{R}^+$. We pick up a

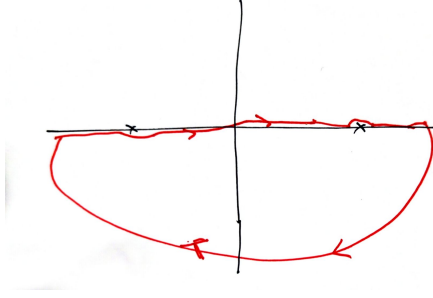
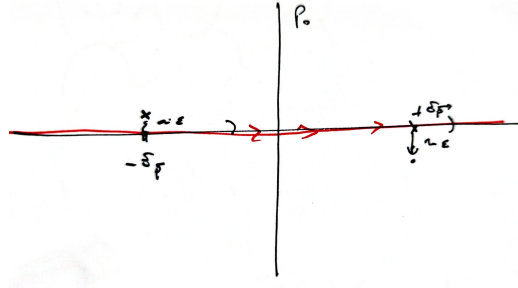


Figure 4: Closing the Feynman contour.

Figure 5: The $i\varepsilon$ version of the Feynman contour.

minus because of contour-orientation:

$$\begin{aligned}
 I_F(x-y) &= \int \frac{d^3p}{(2\pi)^4} i e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \times (-2\pi i) \text{Res}_{p^0 = \pm E_{\vec{p}}} [J(p_0)] \\
 &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \frac{1}{2E_{\vec{p}}} e^{-iE_{\vec{p}}(x^0 - y^0)} \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip \cdot (x-y)} = D(x-y),
 \end{aligned}$$

for $x^0 < y^0$, in the upper half-plane we get a contribution from the pole at $p_0 = -E_{\vec{p}}$. Again by Cauchy's Theorem and Jordan's Lemma, we get that $I_F(x-y) = D(x-y)$, hence $I_F(x-y) = \Delta_F(x-y)$. $\square$

The alternative representation for the contour and the integrand is the " $i\varepsilon$ prescription", where $0 < \varepsilon \ll 1$. Once the poles have been shifted off the real axis, as shown in Figure 5, we can just integrate over the real axis. This gets us

$$\Delta_F(x-y) = \lim_{\varepsilon \rightarrow 0^+} \left[\int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\varepsilon} \right]. \quad (30)$$

5.6 Wick's Theorem

Recall that we're interested computing expectation values of time-ordered strings of operators. Denote $\hat{\varphi}_i = \hat{\varphi}(x_i)$, for some $i = 1, \dots, n$.

Definition 5.8. The **time-ordered product** of the $\widehat{\varphi}_i$ is defined as

$$T_n := T[\widehat{\varphi}_1 \cdots \widehat{\varphi}_n].$$

Recall that for $n = 2$, we derived that

$$T_2 = T[\widehat{\varphi}_1 \widehat{\varphi}_2] =: \widehat{\varphi}_1 \widehat{\varphi}_2 : + \Delta_F(x_1 - x_2).$$

Definition 5.9 (Contraction). Define a **contraction of a string of operators**

$$\overline{\widehat{\varphi}_1 \widehat{\varphi}_2 \widehat{\varphi}_3 \widehat{\varphi}_4} := \Delta_F(x_1 - x_3) \widehat{\varphi}_2 \widehat{\varphi}_4,$$

by underlining notation. Then the sum over all contractions will be denoted $C[\widehat{\varphi}_1 \cdots \widehat{\varphi}_n]$.

Example 5.10. We have

$$C[\widehat{\varphi}_1 \widehat{\varphi}_2 \widehat{\varphi}_3 \widehat{\varphi}_4] := \widehat{\varphi}_1 \widehat{\varphi}_2 \widehat{\varphi}_3 \widehat{\varphi}_4 + [\overline{\widehat{\varphi}_1 \widehat{\varphi}_2 \widehat{\varphi}_3 \widehat{\varphi}_4} + \cdots] + [\overline{\widehat{\varphi}_1 \widehat{\varphi}_2 \widehat{\varphi}_3 \widehat{\varphi}_4} + \cdots].$$

Theorem 5.11 (Wick). For every $n \in \mathbb{N}$, and strings of operators $\widehat{\varphi}_i$, $1 \leq i \leq n$, we have

$$T[\widehat{\varphi}_1 \cdots \widehat{\varphi}_n] =: C[\widehat{\varphi}_1 \cdots \widehat{\varphi}_n] :.$$

Proof. Induct on n . For $n = 2$, we have verified that

$$\begin{aligned} T_2 &=: \widehat{\varphi}_1 \widehat{\varphi}_2 : + \Delta_F(x_1 - x_2) \\ &=: \widehat{\varphi}_1 \widehat{\varphi}_2 : + \overline{\widehat{\varphi}_1 \widehat{\varphi}_2} =: C[\widehat{\varphi}_1 \widehat{\varphi}_2] :. \end{aligned}$$

Assume Wick's Theorem holds for all $l \leq n - 1$. We wish to prove it for $l = n$. Define

$$T_n := T[\widehat{\varphi}_1 \cdots \widehat{\varphi}_n],$$

and assume WLOG that $x_1^0 \geq x_i^0$ for all other i , as we can always re-order inside the time-ordering symbol. Now by inductive assumption, we can do

$$\begin{aligned} T_n &= \widehat{\varphi}_1 T[\widehat{\varphi}_2 \cdots \widehat{\varphi}_n] \\ &= (\widehat{\varphi}_1^+ + \widehat{\varphi}_1^-) : C[\widehat{\varphi}_2 \cdots \widehat{\varphi}_n] : \\ &=: \widehat{\varphi}_1^- C[\widehat{\varphi}_2 \cdots \widehat{\varphi}_n] : + : \widehat{\varphi}_1^+ C[\widehat{\varphi}_2 \cdots \widehat{\varphi}_n] : + : \sum_{i=2}^n [\widehat{\varphi}_1^+, \widehat{\varphi}_i] C[\widehat{\varphi}_2 \cdots \widehat{\varphi}_i \cdots \widehat{\varphi}_n] :. \end{aligned}$$

But by our earlier calculation, we know that $[\widehat{\varphi}_1^+, \widehat{\varphi}_i] = [\widehat{\varphi}_1^+, \widehat{\varphi}_i^-] = \Delta_F(x_1 - y_i)$. It just suffices to now to argue that

$$\begin{aligned} T_n &=: \widehat{\varphi}_1 C[\widehat{\varphi}_2 \cdots \widehat{\varphi}_n] : + \sum_{i=2}^n : C[\overline{\widehat{\varphi}_1 \cdots \widehat{\varphi}_i \cdots \widehat{\varphi}_n}] : \\ &=: \widehat{\varphi}_1 C[\widehat{\varphi}_2 \cdots \widehat{\varphi}_n] + \sum_{i=2}^n C[\overline{\widehat{\varphi}_1 \cdots \widehat{\varphi}_i \cdots \widehat{\varphi}_n}] =: C[\widehat{\varphi}_1 \cdots \widehat{\varphi}_n] :. \end{aligned}$$

□



Figure 6: Representation of contraction.

$$1 + \underbrace{\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots}_{\mathcal{O}(\lambda^2)}$$

The diagram shows the expansion of a vacuum expectation value. It starts with 1, followed by a term proportional to λ (a tadpole diagram), then a bracketed sum of terms proportional to λ^2 (a self-energy loop and a tadpole with a loop), and so on. The terms are labeled $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$.

Figure 7: The expansion in terms of Feynman diagrams.

5.7 From Wick's Theorem to Feynman's Rules

To show the computational power of Wick's Theorem, we will compute in the φ^4 -theory the quantity

$$\begin{aligned} \langle 0 | \hat{S} | 0 \rangle &= \langle 0 | T \left[\exp \left(\frac{1}{i} \frac{\lambda}{4!} \int d^4x \hat{\varphi}^4(x) \right) \right] | 0 \rangle \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{1}{i} \frac{\lambda}{4!} \right)^l \int d^4x_1 \cdots \int d^4x_l \times \mathcal{M}(x_1, \dots, x_l), \end{aligned}$$

where

$$\begin{aligned} \mathcal{M} &= \langle 0 | T [\hat{\varphi}^4(x_1) \cdots \hat{\varphi}^4(x_l)] | 0 \rangle \\ &= \langle 0 | : C [\hat{\varphi}^4(x_1) \cdots \hat{\varphi}^4(x_l)] : | 0 \rangle, \end{aligned}$$

which is non-zero only if all fields are contracted.

In second order, this looks like

$$\mathcal{M}(x_1, x_2) = \langle 0 | \overbrace{\hat{\varphi}(x_1) \hat{\varphi}(x_1)} \overbrace{\hat{\varphi}(x_1) \hat{\varphi}(x_1) \hat{\varphi}(x_2) \hat{\varphi}(x_2)} \overbrace{\hat{\varphi}(x_2) \hat{\varphi}(x_2) \hat{\varphi}(x_2) \hat{\varphi}(x_2)} | 0 \rangle.$$

We can represent these contractions as in Figure . This is the Feynman diagram expansion of $\mathcal{M}$. To do this in generality, we need the Feynman rules to be able to convert the diagram into mathematical expressions.

- (i) For a 4-way crossing x_1 , we add a factor of $-i\lambda$,
- (ii) For a point-to-point from x_1 to x_2 , we add a factor of $\Delta_F(x_1 - x_2)$,
- (iii) We then integrate over each spacetime point,
- (iv) Take care of combinatorial factors.

We will not discuss (iv) here, as it is mostly a concern for loop-level diagrams, which we'll cover in AQFT.

Thus we have the series

$$\langle 0 | \mathcal{S} | 0 \rangle = \sum_{l=0}^{\infty} \text{all vacuum diagrams with } l \text{ vertices},$$

being give by the diagrammatic expansion depicted in Figure 7. Clearly the diagrams with loops will introduce UV divergences, as we can see that loop diagrams translate into $\Delta_F(0)$.

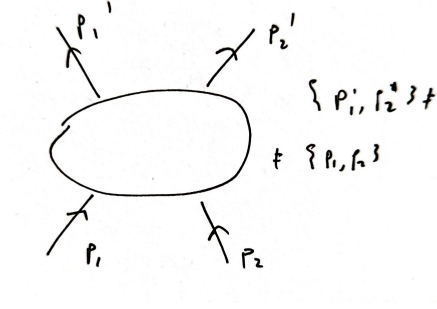


Figure 8: Momentum not being conserved.

5.8 Scattering in Scalar Yukawa Theory

φ^4 is too complex, so we will first study scattering in the Scalar Yukawa Theory. This is a theory of one complex scalar field and one real scalar field. Our real scalar field will be φ , our complex one ψ , with masses m and M respectively.

The Lagrangian will be given by the sum

$$\begin{aligned}\mathcal{L} &= (\partial_\mu \psi^*)(\partial^\mu \psi) - M^2 \psi^* \psi \\ &+ \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2 - g \varphi \psi^* \psi.\end{aligned}$$

The nice thing about this Lagrangian is that it preserves the $U(1)$ -symmetry, which is given by $\psi \rightarrow e^{i\alpha} \psi$, $\psi^* \rightarrow e^{-i\alpha} \psi^*$. Note that we have $[g] = 1$, so to have relevant interaction.

The interaction picture field operators are given by

$$\begin{aligned}\hat{\varphi}(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_{\vec{p}}}} (\hat{a}_{\vec{p}} e^{-ip \cdot x} + \hat{a}_{\vec{p}}^\dagger e^{ip \cdot x}) \\ \hat{\psi}(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (\hat{b}_{\vec{p}} e^{-ip \cdot x} + \hat{c}_{\vec{p}}^\dagger e^{ip \cdot x}) \\ \hat{\psi}^\dagger(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (\hat{b}_{\vec{p}}^\dagger e^{ip \cdot x} + \hat{c}_{\vec{p}} e^{-ip \cdot x}),\end{aligned}$$

where $\mathcal{E}_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$ and $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + M^2}$. This allows us to derive that the only non-trivial commutators are

$$[\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}^\dagger] = [\hat{b}_{\vec{p}}, \hat{b}_{\vec{q}}^\dagger] = [\hat{c}_{\vec{p}}, \hat{c}_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}).$$

Hence the vacuum state for the free theory with $g = 0$ is given by for all $\vec{p} \in \mathbb{R}^3$

$$\begin{aligned}\hat{a}_{\vec{p}}|0\rangle &= \hat{b}_{\vec{p}}|0\rangle = \hat{c}_{\vec{p}}|0\rangle \\ \langle 0|\hat{a}_{\vec{p}}^\dagger &= \langle 0|\hat{b}_{\vec{p}}^\dagger = \langle 0|\hat{c}_{\vec{p}}^\dagger = 0.\end{aligned}$$

Thus we have particles of the ψ field given by

$$\sqrt{2E_{\vec{p}}} \hat{b}_{\vec{p}}^\dagger |0\rangle, \sqrt{2E_{\vec{p}}} \hat{c}_{\vec{p}}^\dagger |0\rangle$$

which both have energy $E = E_{\vec{p}}$, $Q = -1$, $Q = +1$. We will consider now a scattering experiment in which initial and final momenta are not equal, as depicted in Figure .

We set initial and final states as

$$\begin{aligned} |i\rangle &= \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} \hat{b}_{\vec{p}_1}^\dagger \hat{b}_{\vec{p}_2}^\dagger |0\rangle \\ |f\rangle &= \sqrt{2E_{\vec{p}_1'}} \sqrt{2E_{\vec{p}_2'}} \hat{b}_{\vec{p}_1'}^\dagger \hat{b}_{\vec{p}_2'}^\dagger |0\rangle, \end{aligned}$$

hence we want to compute for $\hat{S} = T \left[\exp \left(\frac{g}{i} \int d^2x \hat{\varphi} \hat{\psi}^\dagger \hat{\psi} \right) \right]$

$$\mathcal{A}_{i \rightarrow f} = \langle f | \hat{S} | i \rangle \propto \langle 0 | \hat{b}_{\vec{p}_1'} \hat{b}_{\vec{p}_2'} (\dots) \hat{b}_{\vec{p}_1}^\dagger \hat{b}_{\vec{p}_2}^\dagger | 0 \rangle,$$

where $(\dots)$ needs $\hat{\psi} \hat{\psi} \hat{\psi}^\dagger \hat{\psi}^\dagger$ to be non-zero. Hence we want to consider our expansion of the exponential up to second order $\mathcal{O}(g^2)$, as that will be our leading order expansion. Hence

$$\mathcal{A}_{i \rightarrow f}^{(2)} = \frac{1}{2} \left(\frac{g}{i} \right)^2 \int d^4x_1 \int d^4x_2 \widetilde{\mathcal{M}}(x_1, x_2) \quad (31)$$

$$\widetilde{\mathcal{M}}(x_1, x_2) = \langle f | T \left[\overline{\hat{\varphi}(x_1) \hat{\psi}^\dagger(x_1) \hat{\psi}(x_1) \hat{\varphi}(x_2) \hat{\psi}^\dagger(x_2) \hat{\psi}(x_2)} \right] | i \rangle \quad (32)$$

$$= \Delta_F^{(\psi)}(x_1 - x_2) \langle f | : \hat{\psi}^\dagger(x_1) \hat{\psi}(x_1) \hat{\psi}^\dagger(x_2) \hat{\psi}(x_2) : | i \rangle. \quad (33)$$

Remembering that only the $\hat{b}\hat{b}^\dagger$ oscillators contribute in the mode expansion, we can immediately resolve the normal ordering:

$$\begin{aligned} \widetilde{\mathcal{M}}(x_1, x_2) &= \Delta_F^{(\varphi)}(x_1 - x_2) \times \mathcal{N} \\ \mathcal{N} &:= \langle f | \hat{\psi}^\dagger(x_1) \hat{\psi}^\dagger(x_2) \times \mathbf{1} \times \hat{\psi}(x_1) \hat{\psi}(x_2) | i \rangle, \end{aligned}$$

by the usual trick of insuring a resolution of identity expressed as a sum of a complete set of eigenstates of the free Hamiltonian $\hat{H}_0$

$$\mathbf{1} = \sum_{|\psi\rangle \in \mathcal{B}} |\psi\rangle \langle \psi| = |0\rangle \langle 0| + \sum_{n \geq 1} \sum_{|\psi\rangle \in \mathcal{B},} |\psi\rangle \langle \psi|.$$

But then we note that only contributions from the vacuum state matter, hence going back to $\mathcal{N}$ we can write

$$\begin{aligned} \mathcal{N} &= \langle f | \hat{\psi}^\dagger(x_1) \hat{\psi}^\dagger(x_2) \times \mathbf{1} \times \hat{\psi}(x_1) \hat{\psi}(x_2) | i \rangle \\ &= \langle f | \hat{\psi}^\dagger(x_1) \hat{\psi}^\dagger(x_2) \times |0\rangle \langle 0| \times \hat{\psi}(x_1) \hat{\psi}(x_2) | i \rangle, \end{aligned}$$

which we decompose as $\mathcal{N} = \mathcal{N}_i \mathcal{N}_f$ where

$$\begin{aligned} \mathcal{N}_i &= \langle 0 | \hat{\psi}(x_1) \hat{\psi}(x_2) | i \rangle \\ \mathcal{N}_f &= \langle f | \hat{\psi}^\dagger(x_1) \hat{\psi}^\dagger(x_2) | 0 \rangle \end{aligned}$$

Lecture 15

We have

$$\mathcal{N}_i = \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{q}}}} e^{-ip \cdot x_1 - iq \cdot x_2} \langle 0 | \hat{b}_{\vec{p}} \hat{b}_{\vec{q}} \hat{b}_{\vec{p}}^\dagger \hat{b}_{\vec{q}}^\dagger | 0 \rangle.$$

Applying the commutation relations, we get that

$$\langle 0 | \hat{b}_{\vec{p}} \hat{b}_{\vec{q}} \hat{b}_{\vec{p}}^\dagger \hat{b}_{\vec{q}}^\dagger | 0 \rangle = (2\pi)^6 (\delta^{(3)}(\vec{p} - \vec{p}_1) \delta^{(3)}(\vec{q} - \vec{p}_2) + \delta^{(3)}(\vec{p} - \vec{p}_2) \delta^{(3)}(\vec{q} - \vec{p}_1),$$

which we can reinsert into the integrals to kill them. This reduces to

$$\begin{aligned}\mathcal{N}_i &= (e^{-ip_1 \cdot x_1 - ip_2 \cdot x_2} + e^{-ip_1 \cdot x_2 - ip_2 \cdot x_1}) \\ \mathcal{N}_f &= (e^{+ip'_1 \cdot x_1 + ip'_2 \cdot x_2} + e^{+ip'_1 \cdot x_2 + ip'_2 \cdot x_1}).\end{aligned}$$

Hence reinserting this into Equation 33, we get

$$\begin{aligned}\mathcal{A}_{i \rightarrow f}^{(2)} &= \frac{1}{2} \left(\frac{g}{i}\right)^2 \int d^4x_1 d^4x_2 \mathcal{N}_i \mathcal{N}_f \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik \cdot (x_1 - x_2)} \\ &= \frac{1}{2} \left(\frac{g}{i}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \int d^4x_1 \int d^4x_2 \\ &\quad \left(e^{ix_1 \cdot (-k + p'_1 - p_2)} e^{ix_2 \cdot (k + p'_2 - p_2)} + e^{ix_1 \cdot (-k + p'_2 - p_1)} e^{ix_2 \cdot (k + p'_1 - p_2)} + \text{same with swapped } x_1, x_2 \right).\end{aligned}$$

We now use

$$\int d^4x e^{+ip \cdot x} = (2\pi)^4 \delta^{(4)}(p).$$

This gives

$$\begin{aligned}\mathcal{A}_{i \rightarrow f}^{(2)} &= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} (2\pi)^8 [\delta^{(4)}(-k + p'_1 - p_1) \cdot \delta^{(4)}(k - p'_2 - p_2) \\ &\quad + \delta^{(4)}(-k + p'_2 - p_1) \cdot \delta^{(4)}(k - p'_1 - p_2)] \\ &= i(-ig)^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) \times \left[\frac{1}{(p_1 - p'_1)^2 - m^2 + i\varepsilon} + \frac{1}{(p_1 - p'_2)^2 - m^2 + i\varepsilon} \right],\end{aligned}$$

which is non-zero only when $p'_1 + p'_2 = p_1 + p_2$. This gives energy-momentum conservation, but we still need to discuss the $\varepsilon \rightarrow 0$ momentum. This gives $\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2$, and moreover $E_{\vec{p}_1} + E_{\vec{p}_2} = E_{\vec{p}'_1} + E_{\vec{p}'_2}$, where $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + M^2}$. Since our answer is Lorentz invariant, we can move to the COM frame, where

$$\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2 = 0,$$

which implies $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}'_1| = |\vec{p}'_2|$, and thus $E_{\vec{p}_1} = E_{\vec{p}_2} = E_{\vec{p}'_1} = E_{\vec{p}'_2}$.

Moreover, since $(p_1 - p'_1)^M = (0, \vec{p}_1 - \vec{p}'_1)$, hence $(p_i - p'_i)^2 \leq 0$, hence this is a space-like vector. By reducing the same to the other vector, we can show that it is safe to take the limit $\varepsilon \rightarrow 0$. This shows we can get

$$\mathcal{A}_{i \rightarrow f}^{(2)} = i(-ig)^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) \times \left[\frac{1}{(p_1 - p'_1)^2 - m^2} + \frac{1}{(p_1 - p'_2)^2 - m^2} \right]. \quad (34)$$

5.9 Feynman Rules for scalar Yukawa Theory

Once we did the computation in this hard way, we can develop a graphical calculus, i.e. the Feynman rules, that will allow us to replicate the computation more cleanly.

The Feynman diagrams will be given by the following:

Claim. Non-trivial contributions to $\langle f | \hat{S} - \mathbf{1} | i \rangle$ are in bijection with Feynman diagrams.

Proof. We get the Feynman diagrams as in Figure 9 by



Figure 9: Lines and vertices for Feynman diagrams.

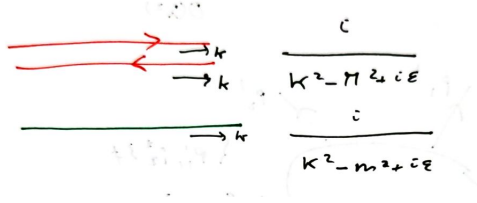


Figure 10: Feynman rules for lines.

- (i) Choose a direction for time.
- (ii) Draw an **external line** for each particle in $|i\rangle$ and each in $|f\rangle$, labelled by their on-shell momenta,
- (iii) Connect external lines with vertex,
- (iv) The **internal lines** connect the vertices.

We then use the Feynman rules to associate each Feynman diagram to a contribution.

- (i) We assign propagators to internal lines, where we allow k^2 to be off-shell momenta, i.e. their 0th component is an independent DoF, as in Figure 10,
- (ii) We assign to each vertex a $\delta^{(4)}$ function enforcing momentum conservation, as in Figure 11,
- (iii) Integrate all internal momenta with $\int \frac{d^4 t}{(2\pi)^4}$, which soaks up all the momenta-conservation delta-functions.

These are called momentum-space Feynman rules, and can be thought as a Fourier-transformed version of the old Feynman rules. $\square$

Example 5.12. Our earlier scattering experiment is depicted as in Figure 12. The leading order diagrams will have four external lines. There are only two topologically

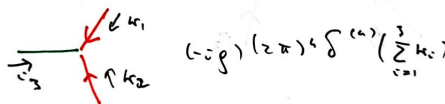


Figure 11: Feynman rules for vertices.

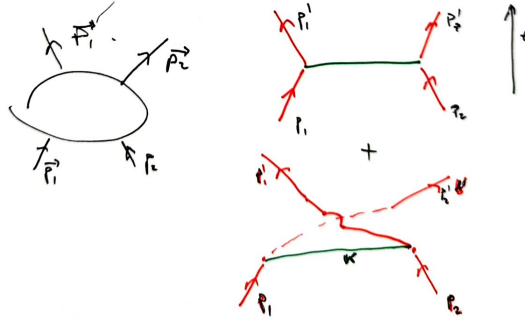


Figure 12: The scattering experiment as a Feynman diagram calculation.

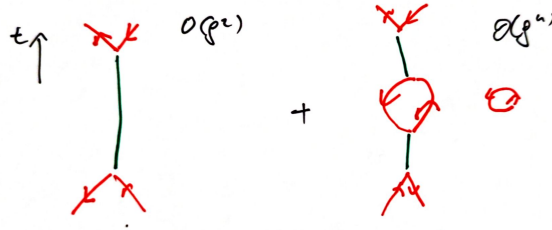


Figure 13: Particle and anti-particle scattering.

distinct Feynman diagrams up to scattering order. This gives

$$\mathcal{A}_{i \rightarrow f}^{(4)} = (-ig)^2 \cdot \left[\frac{i}{(p_1 - p'_1)^2 - m^2} + \frac{i}{(p_1 - p'_2)^2 - m^2} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2).$$

These kinds of calculations will always produce an overall δ -term which enforces momentum preservation.

Lecture 16

Last lecture we developed specific Feynman rules specifically for the scalar Yukawa theory and for the specific class of observables we care about, i.e. scattering amplitude.

Let us now compute the scattering of a particle and anti-particle pair $\psi + \bar{\psi} \rightarrow \psi + \bar{\psi}$. The Feynman diagram up to leading order looks like Figures 13 and 14. At order $\mathcal{O}(g^4)$, we need to discuss which diagrams are allowed and which are not, a loop issue. For example we may get a Fermion loop.

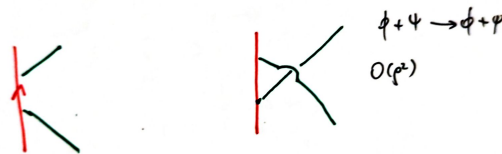


Figure 14: Particle and scalar scattering.

6 Fermions

We will look at a way to derive the Feynman rules, i.e. derive the rules for just the vertices, once we understand the propagators for certain fields, directly from the Hamiltonian.

This saturates our discussion for scalar fields. We will then move to particles with spin. We will first discuss Fermions, and then Bosonic fields.

Recall that a scalar field transforms under a LT by

$$\varphi(x) \xrightarrow{\Lambda} \varphi(\Lambda^{-1} \cdot x),$$

which after quantisation produces a $\hat{\varphi}$, which has a mode expansion which creates particles of spin 0.

Let us recall the non-relativistic description of spin in QM. We model spin by introducing a spin operator $\hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$, modelled on the old theory of angular momentum. The total spin is then given by $\hat{S}$, and we may take particles of spin half integers $s = 0, \frac{1}{2}, 1, \dots$, which gives $(2s + 1)$ spin states, i.e.

$$\hat{S}^2 |s, s_z\rangle = s(s + 1)\hbar |s, s_z\rangle.$$

Now the most familiar case is for spin $\frac{1}{2}$ which is $|\uparrow\rangle, |\downarrow\rangle$. Mathematically, we want to see this as an irreducible representation of the rotation group, or rather for the covering group of $SU(2)$.

To move to the relativistic case, we want to think about representations of rotation groups in terms of sub-representation of non-trivial representation of Lorentz groups. In other words, we need spins transforming in non-trivial representations of G_L .

Recall that G_L being a linear Lie group we may realise it as a group of matrices $\Lambda \in \text{Mat}_4(\mathbb{R})$ such that $\Lambda \cdot \eta \cdot \Lambda^\top = \eta, \det \Lambda = +1$.

Definition 6.1 (Representation). A **representation** D of G_L of dimension N is an assignment

$$D : G \longrightarrow \text{Mat}_N(\mathbb{C}),$$

such that $\det(D(g)) \neq 0$, D is smooth and is a group homomorphism, i.e. $D(\Lambda_1) \cdot D(\Lambda_2) = D(\Lambda_1 \cdot \Lambda_2)$.

These matrices $D(\Lambda)$ act on the vector space $\mathbb{C}^N$ by multiplication.

We say a field

$$\Psi : \mathbb{R}^{3,1} \longrightarrow \mathbb{C}^N,$$

transforms in a representation D of G_L if

$$\Psi_A(x) \xrightarrow{\Lambda \in G_L} \Psi'_A(x) = \sum_{B=1}^N D(\Lambda)_{AB} \Psi_B(\Lambda^{-1} \cdot x).$$

Example 6.2 (Scalar Field). Scalar fields $\varphi(x) \xrightarrow{\sim} \varphi(x^{-1})$ transforms under the trivial representation, i.e. $N = 1$ and $D[\Lambda] = 1$.

Example 6.3 (4-Vector Field). A 4-vector field

$$A^\mu(x) \xrightarrow{\Lambda} \Lambda^\mu_\nu A^\nu(\Lambda^{-1} \cdot x),$$

transforms under the **fundamental representation** given by $N = 4$ and $D[\Lambda] = \Lambda$. This will eventually give spin-1 fields.

Question. How to construct a new representation of the Lorentz group to give spin- $\frac{1}{2}$ particles?

Answer. Use the fact that G_L is a Lie group, and construct the representation on the Lie algebra $\mathbb{L}(G_L)$, and then exponentiate through

$$\exp : \mathbb{L}(G_L) \longrightarrow G_L.$$

This is how we will construct the spinor representation of G_L .

Warning. The exponential map need not be injective or surjective.

Definition 6.4. A **representation of a Lie algebra** $\mathbb{L}(G_L)$ is a linear map

$$R : \mathbb{L}(G_L) \longrightarrow \text{Mat}_N(\mathbb{C}),$$

that preserves the bracket, i.e.

$$\forall X, Y \in \mathbb{L}(G_L). R([X, Y]) = [R(X), R(Y)].$$

Suppose now that we can parametrise the elements of the Lie group, at least in a neighbourhood of the identity by exponentials $\Lambda = \exp(X)$, $X \in \mathbb{L}(G_L)$. Then if R is a representation of $\mathbb{L}(G_L)$ then $D(\Lambda) = \exp(R(X))$ is a representation for G .

Remark. Lots of Lie groups have the same Lie algebras, so D yields a representation of the universal cover of $\widetilde{G}_L$ of G , i.e.

$$e : \widetilde{G}_L \longrightarrow G_L,$$

which is globally 2:1. This is simply connected for simple Lie groups.

Example 6.5 (s). For $SU(2)$, we get $SU(2) \xrightarrow{2:1} SO(3)$. This is immensely important since $G_L = SO(3, 1)^+ \leftarrow \widetilde{G}_L \cong SL(2, \mathbb{C})$, as that is, remarkably, what the quantum theory really sees.

Proper Lorentz transforms lie in the group $G_L = SO(3, 1)$, for $\Lambda = \exp(\omega) \in G_L$, we get an anti-symmetry condition

$$\begin{aligned} \Lambda \in G_L &\implies \Lambda \cdot \eta \cdot \Lambda^\top = \eta \\ &\implies \omega^{\mu\nu} + \omega^{\nu\mu} = 0. \end{aligned}$$

Hence our Lie algebra will be

$$\mathbb{L}(G_L) = \{\omega \in \text{Mat}_4(\mathbb{R}) | \omega + \omega^\top = 0\}.$$

This has a basis of $6 = (4 \times 3)/2$ generators, which we label as

$$\mathcal{M}^{\rho\sigma} = -\mathcal{M}^{\sigma\rho},$$

where $(\mathcal{M}^{\rho\sigma})_{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$. With this basis, we can write a general element of $\mathbb{L}(G_L)$ as a linear combination given by

$$\omega_\nu^\mu = \frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})_\nu^\mu,$$

where we may take w.l.o.g. $\Sigma_{\rho\sigma} = -\Sigma_{\sigma\rho}$ to be anti-symmetric. To understand the bracket, we may just specify it between any basis elements, which will give the structure constants of the Lie algebras. Formally

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}. \quad (35)$$

Lecture 17

Correction: for the particle anti-particle scattering, we could also draw another tree-level diagram, in which the two vertices lie horizontally.

To construct spinor fields, we will need non-trivial representations for the Lorentz group. Last time we wrote an explicit basis for the Lie algebra of the Lorentz group. Having picked this basis, we can pin-down the Lie bracket by specifying it for the generators, and the rest is determined by linearity. This is given by Equation 35. We can use the Lie algebra to reconstruct the Lie group itself, through the exponential map in a neighbourhood of the identity. Taking a proper orthochronous $\Lambda \in SO(3, 1)$, i.e. one in the connected component of $\mathbf{1}$, we can express it in terms of $\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma})$.

Of course, general LTs will correspond to various combinations of rotations and boost, but we can work with this formal in the specific case of rotations, i.e. where the space-like components vanish, i.e. $\Omega_{00} = \Omega_{0i} = \Omega_{i0} = 0$. By anti-symmetry, we can parametrise Ω by

$$\Omega_{ij} = -\varepsilon_{ijk}\varphi^k,$$

where $\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$ is the rotation 3-vector.

Exercise. Setting $\varphi^1 = \varphi^2 = 0$, show that $\Lambda = \exp(-\varphi^2\mathcal{M}^{12})$ acts on a 4-vector as a rotation about the z-axis with angle φ .

Let us recall that our goal is to define fermionic fields. From the non-relativistic case, we know that their distinguishing feature is the property that a rotation of π in space does not bring you back to where you started, but you pick up a factor of minus one. We will construct a new representation of the Lorentz group that is suitable to describe this half integer spin situation, and then go back and check that it has this property. The way we will proceed is by a trick used by Dirac himself, which is to first develop a representation for a simpler algebra, called the **Clifford algebra**.

6.1 The Clifford Algebra

We will mostly treat it as an auxiliary object for the moment.

Definition 6.6 (Clifford Algebra). The **Clifford algebra** has four generators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ satisfying commutation relations

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \cdot \mathbf{1}.$$

It admits no representation in dimension less than 4. Recall that if we have **Pauli matrices**

$$\vec{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

i.e. $\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + \varepsilon_{ijk} \sigma_k$, which obeys $\{\sigma^i, \sigma_j\} = 2\delta^{ij} \mathbf{1}$.

We can get the smallest non-trivial representation of the Clifford algebra by setting

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & 0 \\ -\sigma^i & 0 \end{pmatrix}. \end{aligned}$$

Indeed, we can check that

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}\mathbf{1}.$$

This produces

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho}.$$

▮ **Exercise.** Check this.

This shows that this gets us a representation for Lorentz algebra, hence we call it the **spinor representation**, which is really a representation of the simply-connected cover $G_L \cong SL(2, \mathbb{C})$ which maps 2:1 onto $G_L = SO(3, 1)$. The representation is given for $\Lambda \in G_L$, $\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma})$ maps to $S[\Lambda] \in \text{Mat}_4(\mathbb{C})$ given by $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$.

Example 6.7 (Rotations). In the specific case of rotation, given by $\Lambda = \exp(-\rho^3\mathcal{M}^{12})$. Let us work out what the representation matrix in the spinor representation is for these elements. Recall that we have

$$\begin{aligned} S^{ij} &= \frac{1}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sigma^i\sigma^j \\ &= \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} \\ &= -\frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \end{aligned}$$

Remembering that $\Sigma_{00} = \Sigma_{0i} = \Sigma_{i0} = 0$, we get

$$\begin{aligned} S[\Lambda] &= \exp\left(\frac{1}{2}\Sigma_{ij}S^{ij}\right) \\ &= \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix}. \end{aligned}$$

We can simplify this even further in the case $\varphi^1 = \varphi^2 = 0$, where we get

$$S[\Lambda] = \text{diag}(e^{i\rho^3/2}, e^{-i\rho^3/2}, e^{i\rho^3/2}, e^{-i\rho^3/2}).$$

This shows our key feature of our representation obeys $S[\varphi^3 = 2\pi] = -\mathbf{1}_4$, but we have $\Lambda[\varphi^3 = 2\pi] = \mathbf{1}_4$. Hence we have really produced a representation of the covering group!

Remark. If you approached them by classifying representations of the Lorentz group, you would discover that the spin representation is the essentially only other four dimensional representation that you can use to get the Lorentz group.

The big benefit of this is that now we can define fields with respect to this representation.

Definition 6.8 (Spinor Field). A **spinor field** a function

$$\begin{aligned}\psi : \mathbb{R}^{3,1} &\longrightarrow \mathbb{C}^4 \\ x &\longmapsto \psi^\alpha(x) \in \mathbb{C},\end{aligned}$$

where $\alpha \in \{1, \dots, 4\}$ is called a **spinor index**, which transforms under Lorentz transform $\Lambda \in G_L$ as

$$\psi^\alpha(x) \xrightarrow{\Lambda} \psi^\alpha(x) = S[\Lambda]^\alpha_\beta \psi^\beta(\Lambda^{-1} \cdot x), \quad (36)$$

shortened to $\psi(x) \rightarrow S(\Lambda) \cdot \psi(\Lambda^{-1} \cdot x)$.

Our next task will be to construction a Lorentz-invariant action for the spinor field. This will be our natural candidate to describe a spin- $\frac{1}{2}$ field theory.

Recall that, as shown in Example 6.7, a spinor field transforms as $\psi(x) \rightarrow -\psi(x)$ under a 2π rotation. Otherwise, we can proceed to define other related fields in the usual way. For example, the spacetime derivative $\partial_\mu \psi(x)$ will transform as

$$\partial_\mu \psi(x) \longrightarrow (\Lambda^{-1})^\nu_\mu S[\Lambda] \cdot \partial_\nu \psi(\Lambda^{-1} \cdot x).$$

But since the spinor representation is complex, we also need to think about complex conjugation and how the Hermitian conjugate of the spinor field transforms. We'll do that in the next lecture, where we'll find that this will lead us to a simple action, which is the Dirac action for the spinor field.

Lecture 18

The conjugate field transforms under Λ as

$$\psi^\dagger_\alpha(x) \longrightarrow \psi^\dagger_\beta(\Lambda^{-1} \cdot x) (S[\Lambda]^\dagger)^\beta_\alpha.$$

Unfortunately, here we must note that the spinor representation is not unitary, which is quite unusual in physics, as it will violate preservation of norm. This is inevitable, as our Lie group is non-compact, hence by a classical Lie theory result, it admits no finite dimensional unitary representation. Hence, in general, we need not have that $S[\Lambda]^\dagger = S[\Lambda]^{-1}$ for all $\Lambda \in G_L$.

But at the end of the day, the states in the Hilbert space will be given at the end of the day by the modes of this operator, so, as long as those end up being unitary, we'll be fine. We'll also show we're able to get a Lorentz-invariant action that is real, which is a priori not necessarily true.

By the commutator relations for the Clifford algebra, we must have $(\gamma^0)^2 = \mathbf{1}_4$, and thus γ^0 has real eigenvalues, and thus is Hermitian $(\gamma^0)^\dagger = \gamma^0$. This deals with the case $\mu = \nu = 0$, for the other case $\mu = \nu = i$, we have $(\gamma^i)^2 = -\mathbf{1}_4$, γ^i has imaginary

eigenvalues and is anti-Hermitian, with $(\gamma^i)^\dagger = -\gamma^i$. But we can simplify the situation by noting that $(\gamma^0)^3 = \gamma^0 = (\gamma^0)^{-1}$, and moreover that

$$\begin{aligned}\gamma^0 \gamma^i \gamma^0 &= \gamma^0 \{\gamma^i, \gamma^0\} - \gamma^0 \gamma^0 \gamma^i \\ &= -\gamma^i = (\gamma^i)^\dagger,\end{aligned}$$

hence

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \quad (37)$$

We will use Equation 37 to deduce the Hermitian conjugates of $S^{\mu\nu}$:

$$\begin{aligned}(S^{\mu\nu})^\dagger &= \frac{1}{4}([\gamma^\mu, \gamma^\nu])^\dagger \\ &= \frac{1}{4}(([\gamma^\nu]^\dagger, [\gamma^\mu]^\dagger)) \\ &= \frac{1}{4}[\gamma^0 \gamma^\nu \gamma^0, \gamma^0 \gamma^\mu \gamma^0] \\ &= \frac{1}{4}\gamma^0 [\gamma^\nu, \gamma^\mu] \gamma^0 \\ &= -\gamma^0 S^{\mu\nu} \gamma^0.\end{aligned}$$

Let us now consider

$$\begin{aligned}(S[\Lambda])^\dagger &= \exp\left(\frac{1}{2}\Omega\rho\sigma S^{\rho\sigma}\right)^\dagger \\ &= \exp\left(\frac{1}{2}\Omega\rho\sigma (S^{\rho\sigma})^\dagger\right) \\ &= \exp\left(-\frac{1}{2}\Omega\rho\sigma \gamma^0 S^{\rho\sigma} \gamma^0\right) \\ &= \gamma^0 \exp\left(-\frac{1}{2}\Omega\rho\sigma S^{\rho\sigma}\right) \gamma^0 = \gamma^0 S[\Lambda]^{-1} \gamma^0.\end{aligned}$$

Definition 6.9 (Dirac Adjoint). The **Dirac adjoint** is defined as

$$\bar{\psi}(x) := \psi^\dagger(x) \cdot \gamma^0,$$

and transforms as

$$\begin{aligned}\bar{\psi}(x) &\longrightarrow \bar{\psi}'(\Lambda^{-1} \cdot x) S^\dagger[\Lambda] \cdot \gamma^0 \\ &= \psi^\dagger(\Lambda^{-1} \cdot x) \gamma^0 S^{-1}[\Lambda] (\gamma^0 \gamma^0) \\ &= \bar{\psi}(\Lambda^{-1} \cdot x) S[\Lambda]^{-1}.\end{aligned}$$

Define the scalar field

$$\Sigma(x) := \bar{\psi}(x) \psi(x),$$

where the spinor indices are contracted with each other.

Claim. This is a real scalar field.

Proof. Reality follows from

$$\begin{aligned}\Sigma^*(x) &= (\psi^\dagger(x)(\gamma^0)^\dagger\psi(x)) \\ &= \psi^\dagger(x)\gamma^0\psi(x) = \Sigma(x).\end{aligned}$$

To prove scalar field, take some LT $\Lambda \in G_L$,

$$\begin{aligned}\Sigma(x) = \bar{\psi}(x) \cdot \psi(x) &\longrightarrow \Sigma'(x) = \bar{\psi}(\Lambda^{-1} \cdot x) S[\Lambda]^{-1} \cdot S[\Lambda] \psi(\Lambda^{-1} \cdot x) \\ &= \bar{\psi}(\Lambda^{-1} \cdot x) \psi(\Lambda^{-1} \cdot x) = \Sigma(\Lambda^{-1} \cdot x).\end{aligned}$$

□

We need to deal with derivatives before we go on to define the action.

Proposition 6.10. We can construct vector fields by taking

$$V^\mu(x) := \bar{\psi}(x) \gamma^\mu \psi(x), \quad (38)$$

which transforms under Λ as

$$\begin{aligned}V^\mu(x) &\longrightarrow \Lambda^\mu_\nu V^\nu(\Lambda^{-1} \cdot x) \\ &= \Lambda^\mu_\nu \bar{\psi}(\Lambda^{-1} \cdot x) \gamma^\nu \psi(\Lambda^{-1} \cdot x).\end{aligned}$$

Remark. This is the quantity that we need to couple the Maxwell field.

Proof. Under Λ , this transforms as

$$V^\mu(x) \longrightarrow \bar{\psi}(\Lambda^{-1} \cdot x) S[\Lambda]^{-1} \gamma^\mu \cdot S[\Lambda] \psi(\Lambda^{-1} \cdot x),$$

which is equivalent to showing that

$$S[\Lambda]^{-1} \cdot \gamma^\mu \cdot S[\Lambda] = \Lambda^\mu_\nu \gamma^\nu.$$

We'll prove this at the linearised level, by parametrising $\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma})$, and $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma})$. The implied relation is that

$$[\gamma^\mu, S^{\rho\sigma}] = \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho,$$

which is left as an exercise. □

6.2 The Dirac Action

We now have all the ingredients to introduce the Dirac action, which is the only renormalisable action for the fermionic theory. As usual we will demand that this action for spinor fields will be Lorentz invariant and real. Writing the action in terms of the Lagrangian density

$$S = \int d^4x \mathcal{L}(x),$$

this translates to $\mathcal{L}(x)$ is scalar, and (x) is real, at least up to a surface term.

Based on our earlier experience, we could define

$$\mathcal{L} = \partial_\mu \bar{\psi} \partial^\mu \psi,$$

but this is the wrong answer, as it leads to component-wise Klein-Gordon equations, which lead to too many states for the fermionic theory. The correct density to use is

$$\mathcal{L}(x) = \mathcal{L}_1 + \mathcal{L}_2 = \bar{\psi}(x)\gamma^\mu\partial_\mu \cdot \psi(x) - m\bar{\psi}(x)\psi(x),$$

for $m \in \mathbb{R}$, which crucially has a single derivative. This will lead us to Dirac's Equation.

Lecture 19

Claim. $\mathcal{L}(x)$ is a scalar field.

Proof. We have $\mathcal{L}_2 = -m\Sigma(x)$, and $m \in \mathbb{R}$, so we're done. For

$$\mathcal{L}_1(x) = \bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x) \longrightarrow \bar{\psi}(\Lambda^{-1} \cdot x)S[\Lambda]^{-1}i\gamma^\mu \cdot (\Lambda^{-1})^\nu_\mu S[\Lambda]\partial_\nu\psi(\Lambda^{-1} \cdot x),$$

suppressing spinor indices. By Proposition 6.10, we know that this transforms to

$$\begin{aligned} &= \bar{\psi}(\Lambda^{-1} \cdot x)i\Lambda^\mu_\rho\gamma^\rho \cdot (\Lambda^{-1})^\nu_\mu\partial_\nu\psi(\Lambda^{-1} \cdot x) \\ &= i\bar{\psi}(x)(\Lambda^{-1} \cdot x)\gamma^\mu\partial_\mu\psi(\Lambda^{-1} \cdot x) = \mathcal{L}_1(\Lambda^{-1} \cdot x). \end{aligned}$$

□

Remark. This will give us a Lorentz invariant action, as previously discussed.

Claim. $\mathcal{L}(x)$ is real, up to a surface term.

Proof. We already showed that $\mathcal{L} + 2(x) = -m\Sigma(x) \in \mathbb{R}$. For $\mathcal{L}_1$ we have

$$\begin{aligned} \mathcal{L}_1(x) &= i\psi^\dagger(x)\gamma^0 \cdot i\gamma^\mu\partial_\mu\psi(x) \\ \implies \mathcal{L}_1^*(x) &= -i\partial_\mu\psi^\dagger(x)\gamma^0\gamma^\mu\gamma^0 \cdot i\gamma^0\psi(x) \\ &= -i\partial_\mu\bar{\psi}(x) \cdot \gamma^\mu\psi(x) \\ &= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - i\partial_\mu(\bar{\psi}(x)\gamma^\mu\psi(x)) \\ &= \mathcal{L}_1(x) + \text{surface term}. \end{aligned}$$

□

Definition 6.11 (Dirac Action). The **Dirac action** is defined as

$$S = \int d^4x (\bar{\psi}(x) \cdot (i\gamma^\mu\partial_\mu - m) \cdot \psi(x)).$$

Remark. We could apply the principle of least action by varying either ψ or $\bar{\psi}$. We prefer using $\bar{\psi}$, because the action does not contain its derivative. This gets us:

Proposition 6.12 (Dirac's Equation). We have $(i\gamma^\mu\partial_\mu - m) \cdot \psi(x) = 0$.

In a sense, there is no classical spinor field, as they obey the Pauli exclusion principles. So we will start to prepare to quantise the fermionic field. Let us note that by varying w.r.t. $\psi(x)$, we get

$$i(\partial_\mu\bar{\psi}) \cdot \gamma^\mu + m\bar{\psi} = 0.$$

Notation. For any 4-vector V_μ we define $(\hat{V})_{\alpha\beta} = \gamma_{\alpha\beta}^\mu V_\mu$.

Since $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$, for rotations $\Omega_{ij} = -\varepsilon_{ijk}\varphi^k$, we can write out the representative

$$S[\varphi] = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix},$$

and moreover for boosts $\Omega_{i0} = -\Omega_{0i} = \chi_i$, hence

$$S[\vec{\chi}] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

Hence every matrix in the spinor representation is block-diagonal, and the spinor representation is reducible, i.e. it has an invariant subspace.

Recall that our basic strategy for constructing representations of Lie group was to construct ones for the Lie algebra, which only sees in a neighbourhood of the identity. This technique really gives us a representations for the covering group, i.e. $\widetilde{G}_L = \text{Spin}(3,1)$ instead of $G_L = SO(3,1)$, as they both have the same Lie algebra. Something else we could do is that we could complexify, which reduces the problem of dealing with a non-compact group, such as $SO(3,1)$, we get

$$\mathbb{L}_{\mathbb{C}}(SO(3,1)) \cong \mathbb{L}_{\mathbb{C}}(SO(4)) \cong \mathbb{L}_{\mathbb{C}}(SU(2)) \oplus \mathbb{L}_{\mathbb{C}}(SU(2)),$$

and thanks to Cartan's Classification, we know the finite dimensional irreducible representations of $\mathbb{L}_{\mathbb{C}}$. These are given for j half-integer such that $-j \leq j_z \leq +j$, we have the finite dimensional representations R_j which has dimensionality $2j+1$, and $R_{(j_+, j_-)}$ with dimensionality $(2j_+ + 1) \times (2j_- + 1)$. This gives scalar, left-handed and right-handed Weyl spinor and vector spinor. Hence the Dirac spinor representation decomposes as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

6.3 Solutions of Dirac Equation

We see that for

$$(i\gamma^\mu \partial_\mu - m\mathbf{1}_4) \cdot \psi(x) = 0,$$

if we act on it by $(-i\gamma^\nu \partial_\nu - m\mathbf{1}_4)$ and apply the Clifford algebra equations, we get the Klein-Gordon Equation

$$(\partial_\mu \partial^\mu + m^2) \cdot \psi(x) = 0.$$

This backs up the idea that the correct EoM for the spinor fields is the Dirac Equation, which has fewer propagating degrees of freedom than the Klein-Gordon Equation.

Lecture 20

We can see this by taking a Klein-Gordon solution

$$\psi \sim e^{\pm ip \cdot x},$$

with $p_\mu p^\mu = m^2, p^0 = E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$, hence the positive frequency solution is given by

$$\psi_\alpha(x) = u_\alpha(p)e^{-ip \cdot x},$$

which we can plug into Dirac's Equation

$$(p_\mu \gamma^\mu - m \mathbf{1}_4) \cdot u(p) = 0.$$

We now use the chiral representation of γ s in terms of Pauli matrices, and expand the above to

$$\begin{pmatrix} -m \mathbf{1} & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -m \mathbf{1} \end{pmatrix},$$

where $\omega^\mu = (\mathbf{1}_2, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^i)$. Hence the general solution to the mode expansion is given by

$$u(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu \zeta} \\ \sqrt{p_\mu \bar{\sigma}^\mu \zeta} \end{pmatrix},$$

with $\zeta \in \mathbb{C}^2$ is arbitrary. Here the square root of the matrix is meaningful because we have Hermitian operators $H = H^\dagger$, hence we can $H = U^{-1} \Lambda U$, with $\Lambda = \text{diag}\{\lambda_1, \lambda_2\}$, and thus we define $\sqrt{\Lambda} := \text{diag}\{\sqrt{\lambda_1}, \sqrt{\lambda_2}\}$, $\sqrt{H} := U^{-1} \sqrt{\Lambda} U$.

Defining

$$\begin{aligned} \mathcal{M}(p) &:= p_\mu \sigma^\mu, \\ \bar{\mathcal{M}}(p) &:= p_\mu \bar{\sigma}^\mu, \end{aligned}$$

which obey $\mathcal{M}(p) \bar{\mathcal{M}}(p) = m^2 \mathbf{1}_2$, we can check the ansatz.

Choosing a basis $\zeta^s \in \mathbb{C}^2$, $s \in \{1, 2\}$, with $(\zeta^r)^\dagger \cdot \zeta^s = \delta^{rs}$, and similarly $\eta^s \in \mathbb{C}^2$, $(\eta^r)^\dagger \eta^s = \delta^{rs}$, gives us a basis of solutions for positive and negative frequency solutions

$$\begin{aligned} \psi_p^{(\mu)}(x) &= u^{(\mu)}(p) e^{-ip \cdot x}, u^{(\mu)}(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu \zeta^\mu} \\ \sqrt{p_\mu \bar{\sigma}^\mu \zeta^\mu} \end{pmatrix} \\ \bar{\psi}_p^{(\mu)}(x) &= v^{(\mu)}(p) e^{+ip \cdot x}, v^{(\mu)}(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu \eta^\mu} \\ \sqrt{p_\mu - \bar{\sigma}^\mu \eta^\mu} \end{pmatrix}. \end{aligned}$$

Since our spinor mode expansions will involve a number of these operators, we will use some useful identities such as the following ones

$$\begin{aligned} (u_\alpha^r)^\dagger(p) u_\alpha(s) &= 2p_0 \delta^{rs} \\ \sum_{s=1}^2 u_\alpha^s(p) \bar{u}_\beta^s(p) &= (\not{p} + m \mathbf{1}_4)_{\alpha\beta}. \end{aligned}$$

6.4 Quantisation of the Free Spinor Field

Our starting point will be the Dirac action

$$\mathcal{L} = i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - m \bar{\psi}(x) \psi(x).$$

This Lagrangian density has an $U(1)$ symmetry, which we call vector symmetry:

$$\begin{aligned} \psi(x) &\longrightarrow e^{-i\alpha} \psi(x) \\ \bar{\psi}(x) &\longrightarrow e^{+i\alpha} \bar{\psi}(x), \end{aligned}$$

which is also a global symmetry of the action. The associated Noether current

$$\begin{aligned} j_V^\mu &= \bar{\psi} \gamma^\mu \psi = V^\mu \implies \partial_\mu j_V^\mu = 0 \\ Q &= \int d^3x j_V^0 = \int d^3x \bar{\psi} \gamma^0 \psi = \int d^3x \psi^\dagger \psi. \end{aligned}$$

When we quantise, we will be able to get a charge operator. This will also be preserved in the quantum world, in the sense that it commutes with the Hamiltonian, and hence we can simultaneously diagonalise it. This will show that our particles, i.e. eigenstates, will have, other than energy and spin, also an associated charge.

The Hamiltonian formulation is now given as follows. We have

$$\psi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\psi}(x)} = i\vec{\psi}(x)\gamma^0 = i\psi^\dagger(x),$$

hence $\psi_\varphi \sim \dot{\varphi}$. The fact that these degrees of freedom are not independent will be ultimately responsible for the diminished degrees of freedom of the theory.

The Hamiltonian density obtained by Legendre transform will be

$$\begin{aligned}\mathcal{H}(x) &= \psi(x)\dot{\psi}(x) - \mathcal{L}(x) \\ &= \bar{\psi}(x) (-i\gamma^i \partial_i + m) \psi(x).\end{aligned}$$

We now expand the field in a complete set of solutions of Dirac Equation, i.e. take the Fourier transform. Our basis of solutions will be labelled by p_μ , $s = 1, 2$, $p_\mu p^\mu = m^2$, $p_0 > 0$, with

$$\psi_\alpha(x) = \sum_{s=1}^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (b_{\vec{p}}^s u_\alpha^s(p) e^{-ip \cdot x} + c_{\vec{p}}^{s*} v_\alpha^s(p) e^{+ip \cdot x}).$$

Claim. Particles of spin $\frac{1}{2}$ must be quantised as fermions, i.e. we must use anti-commutators instead of commutators.

Lecture 21

Recall that we have

$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \\ \mathcal{H} &= \bar{\psi}(x) (-i\gamma^i \partial_i + m) \psi.\end{aligned}$$

Quantisation, in the Schrödinger picture follows the process

$$\begin{aligned}\psi_\alpha(x) &= \psi_\alpha(\vec{x}, t) \longrightarrow \hat{\psi}_\alpha(\vec{x}), \\ \psi_\alpha^\dagger(x) &\longrightarrow \hat{\psi}_\alpha^\dagger(\vec{x}),\end{aligned}$$

imposing the anti-commutator relations

$$\begin{aligned}\{\hat{\psi}_\alpha(\vec{x}), \hat{\psi}_\beta^\dagger(\vec{y})\} &= \mathbf{1}_{\alpha\beta} \gamma^{(3)}(\vec{x} - \vec{y}) \\ \{\hat{\psi}_\alpha^\dagger(\vec{x}), \hat{\psi}_\beta(\vec{y})\} &= \{\hat{\psi}_\alpha(\vec{x}), \hat{\psi}_\beta(\vec{y})\} = 0.\end{aligned}$$

If we do this, we get particles which obey the Fermi-Dirac statistics, and a vacuum which has a fixed energy rather than a spectrum that is unbounded from below.

We now mode-expand with respect to a complete set of solution of Dirac Equation, and not Klein-Gordon, Our basis of solutions will be labelled by p_μ , $s = 1, 2$, $p_\mu p^\mu = m^2$, $p_0 > 0$. Since we work in Schrödinger picture, time is fixed for our operators, and hence we have

$$\hat{\psi}_\alpha(\vec{x}) = \sum_{s=1}^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (\hat{b}_{\vec{p}}^s u_\alpha^s(p) e^{-i\vec{p} \cdot \vec{x}} + \hat{c}_{\vec{p}}^{s\dagger} v_\alpha^s(p) e^{-i\vec{p} \cdot \vec{x}}).$$

Exercise. Show that all of the following anti-commutators vanish

$$\begin{aligned}\{\widehat{b}, \widehat{c}\} &= \{\widehat{b}^\dagger, \widehat{c}^\dagger\} \\ &= \{\widehat{b}, \widehat{c}^\dagger\} = \{\widehat{b}^\dagger, \widehat{c}\} = 0,\end{aligned}$$

but we have

$$\{\widehat{b}_{\vec{p}}^s, \widehat{b}_{\vec{q}}^{r\dagger}\} = \{\widehat{c}_{\vec{p}}^s, \widehat{c}_{\vec{q}}^{r\dagger}\} = (2\pi)^3 \delta^{r,s} \delta^{(3)}(\vec{p} - \vec{q}).$$

The point of mode expansion is to write out the Hamiltonian as an infinite set of decoupled harmonic oscillators, and that the Hamiltonian could be thus diagonalised together with those operators. However in this case, we'll have an infinite set of **fermionic SHO**. We have (fermionic) creation and annihilation operators subject to

$$\begin{aligned}\{\widehat{b}, \widehat{b}^\dagger\} &:= \widehat{b}\widehat{b}^\dagger + \widehat{b}^\dagger\widehat{b} = 1 \\ \{\widehat{b}, \widehat{b}\} &= \{\widehat{b}^\dagger, \widehat{b}^\dagger\} = 0.\end{aligned}$$

The second relation shows that these operators are nilpotent

$$\widehat{b}^2 = (\widehat{b}^\dagger)^2 = 0,$$

which is hard to realise for differential operators¹. Hence the Hamiltonian is given by

$$\begin{aligned}\widehat{H}_F &= \frac{1}{2}\omega \left(\widehat{b}^\dagger\widehat{b} - \widehat{b}\widehat{b}^\dagger \right) \\ &= \omega \left(\widehat{N}_F - \frac{1}{2} \right),\end{aligned}$$

where the fermionic number operator is given by $\widehat{N}_F = \widehat{b}^\dagger\widehat{b}$. Note the sign difference in the Hamiltonian compared to the bosonic case.

We have that $\widehat{b}, \widehat{b}^\dagger$ raise and lower energy, since

$$\begin{aligned}[\widehat{H}_F, \widehat{b}] &= -\omega\widehat{b} \\ [\widehat{H}_F, \widehat{b}^\dagger] &= +\omega\widehat{b}^\dagger.\end{aligned}$$

This gives us access to the usual technique to obtain representations of these operators: we find a ground state $|0\rangle$ such that $\widehat{b}|0\rangle = 0 \implies \widehat{N}_F|0\rangle = 0$. This gives

$$\widehat{H}_F|0\rangle = \left(-\frac{1}{2}\omega\right)|0\rangle.$$

The negative sign is the origin of supersymmetry, where fermionic and bosonic field have a tendency to have energies which cancel each other in some way.

We now define

$$\begin{aligned}|1\rangle &:= \widehat{b}^\dagger|0\rangle \\ \implies \widehat{N}_F|1\rangle &= |1\rangle \\ \implies \widehat{H}_F|1\rangle &= \left(+\frac{1}{2}\omega\right)|1\rangle.\end{aligned}$$

But then by nilpotency, that's it, as the next state will square back to $|0\rangle$. Explicitly

$$(\widehat{b}^\dagger)^2|0\rangle = 0.$$

¹**Expand: Is it though?**

Hence $|0\rangle$ and $|1\rangle$ span $\mathcal{H}$, which is an incarnation of the Pauli exclusion principle.

The quantum theory of the spinor field is an infinite decoupled set of fermionic simple Harmonic oscillators, and we will show that in the Example Sheet. We will take the classical Hamiltonian and promote it to an operator

$$H = \int d^3x \mathcal{H} \rightsquigarrow \mathcal{H}.$$

This will get us

$$\hat{H} = \int \frac{d^3x}{(2\pi)^3} E_{\vec{p}} \times \sum_{s=1}^2 \left(\hat{b}_{\vec{p}}^{s\dagger} \hat{b}_{\vec{p}}^s - \hat{c}_{\vec{p}}^s \hat{c}_{\vec{p}}^{s\dagger} \right).$$

We now want to normal order the products to express everything in terms of number operators:

$$\begin{aligned} \hat{H} &= \int \frac{d^3x}{(2\pi)^3} E_{\vec{p}} \times \sum_{s=1}^2 \left(\hat{b}_{\vec{p}}^{s\dagger} \hat{b}_{\vec{p}}^s - \hat{c}_{\vec{p}}^s \hat{c}_{\vec{p}}^{s\dagger} \right) \\ &= \int \frac{d^3x}{(2\pi)^3} E_{\vec{p}} \times \left(\sum_{s=1}^2 \left(\hat{b}_{\vec{p}}^{s\dagger} \hat{b}_{\vec{p}}^s - \hat{c}_{\vec{p}}^s \hat{c}_{\vec{p}}^{s\dagger} \right) - (2\pi)^3 \delta^{(3)}(0) \right) \\ \implies : \hat{H} : &= \int \frac{d^3x}{(2\pi)^3} E_{\vec{p}} \times \sum_{s=1}^2 \left(\hat{b}_{\vec{p}}^{s\dagger} \hat{b}_{\vec{p}}^s - \hat{c}_{\vec{p}}^s \hat{c}_{\vec{p}}^{s\dagger} \right), \end{aligned}$$

by our previous normal ordering discussion. Note that the $\delta^{(3)}(0)$ term corresponds to the sum of the vacuum energy of our fermionic simple harmonic oscillators.

Now if we proceeded with quantising the fields as if they were bosons, the whole thing will go through, except that at the end we'll get a $+$ instead of our $-$. This will give us a Hamiltonian that is unbounded below. This is done in David Tong's notes, and is formally formalised as the Spin-Statistics Theorem.

6.5 Computing the Spectrum

Onto computing the spectrum of this theory. Remember that from our earlier discussion we had shown that the theory had a $U(1)$ -symmetry, with associated Noether currents

$$j_\nu^\mu = \bar{\psi} \gamma^\mu \psi$$

$$Q = \int d^2x \psi^\dagger(\vec{x}) \psi(\vec{x}),$$

which we can quantise as

$$\begin{aligned} \hat{Q} &= \int \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x}) d^3x \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^2 \left(\hat{b}_{\vec{p}}^{s\dagger} \hat{b}_{\vec{p}}^s - \hat{c}_{\vec{p}}^s \hat{c}_{\vec{p}}^{s\dagger} \right), \end{aligned}$$

Now the particle space will be realised as a tensor product of those of the individual fermionic SHOs. We proceed by getting the vacuum state in the particle spectrum by setting all the oscillators to their vacuum state

$$\forall r \in \{1, 2\}, \vec{p} \in \mathbb{R}^3, \hat{b}_{\vec{p}}^r |0\rangle = \hat{c}_{\vec{p}}^r |0\rangle = 0.$$

The 1-particle states, $s \in \{1, 2\}$

$$\hat{b}^{s\dagger} + \vec{p}|0\rangle = |p, s\rangle,$$

where $E = E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$, $Q = +1$. Thus s is labelling the spin states, which we take to be interpreted as $s = 1$ is $\hat{a}, s_z = -\frac{1}{2}$, and $s = 2, \uparrow, s_z = +\frac{1}{2}$. The c -states will correspond to anti-particles

$$\overline{|p, s\rangle} = \hat{c}_{\vec{p}}^{\dagger s} |0\rangle,$$

for $E = E_{\vec{p}}, Q = -1$ ². So a free relativistic massive spin- $\frac{1}{2}$ field theory leads inevitably to the prediction of an anti-particle for any fermion.

The two particle states are given by

$$\begin{aligned} |p, r, q, s\rangle &= \hat{b}_{\vec{p}}^{r\dagger} \hat{b}_{\vec{q}}^{s\dagger} |0\rangle \\ &= -|q, s, p, r\rangle, \end{aligned}$$

as a consequence of $\{\hat{b}_{\vec{p}}^{r\dagger}, \hat{b}_{\vec{q}}^{s\dagger}\} = 0$, which is a manifestation of the Fermi-Dirac statistics.

6.6 Interactive Fermions

We want to deal with interactions, by writing a Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} \\ \mathcal{L}_0 &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \cdot \psi. \end{aligned}$$

Since our perturbative expansions are guided by dimensional analysis, we must appreciate that this is different from the bosonic field in the following

$$[\mathcal{L}] = 4, [\partial] = +1 \implies [\psi] = [\bar{\psi}] = \frac{3}{2}.$$

Thus the first Lorentz-invariant interacting action we could write out

$$\mathcal{L}_{\text{int}} = -g_F (\bar{\psi}\psi)^2,$$

thus the Fermi coupling constant $[g_F] = -2$ is already an irrelevant coupling constant, and thus non-renormalisable. We will see this better in AQFT while studying loop-level corrections.

Thus we can really just consider a Yukawa theory, with a real scalar field

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi) - \frac{\mu^2}{2}\psi^2 - \lambda\psi\bar{\psi}\psi,$$

where the Yukawa coupling $[\lambda] = 0$ gives a marginal coupling.

Next time we'll go onto gauge fields.

Lecture 22

We can again compute scattering amplitudes by using Dyson's formula, then taking the time-ordered products, apply Wick's Theorem, with suitable modification for anti-commutator. Once this is done, Feynman rules and propagator rewrites can be extracted.

²**Todo: Fix all the $s^\dagger$ to $\dagger s$**

Definition 6.13 (Feynma Propagator). Define

$$\begin{aligned} S_F(x-y)_{\alpha\beta} &:= \langle 0 | T[\widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(x) | 0 \rangle \\ &:= \langle 0 | \widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(x) \text{ when } y_0 < x_0 \\ &:= -\langle 0 | \widehat{\bar{\psi}}_\beta(x)\widehat{\psi}_\alpha(x) | 0 \rangle \text{ when } x_0 < y_0, \end{aligned}$$

where the operators are in Heisenberg picture.

Calling the above operators $\mathcal{M}_+, \mathcal{M}_-$, we have

$$\begin{aligned} \mathcal{M}_+ &:= \langle 0 | \widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(x) \\ &= \sum_{r,s=2}^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{q}}}} u_\alpha^s \bar{u}_\beta^r(q) e^{-i(p \cdot x - q \cdot y)} \langle 0 | \{\widehat{b}_{\vec{p}}^r \widehat{b}_{\vec{q}}^{s\dagger}\} | 0 \rangle \\ &= \sum_{r,s=2}^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{q}}}} u_\alpha^s \bar{u}_\beta^r(q) e^{-i(p \cdot x - q \cdot y)} (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (\not{p} + m)_{\alpha\beta} e^{-ip \cdot (x-y)}, \end{aligned}$$

with a similar expression for $\mathcal{M}_-$. This gives an integral representation of S_F

$$S_F(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} e^{-ip \cdot (x-y)}.$$

Theorem 6.14 (Wick for Spinors). We have

$$T[\widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(y)] =: \widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(y) : + \overline{\widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(y)},$$

becomes after taking vacuum expectation values

$$\widehat{\psi}_\alpha(x)\widehat{\bar{\psi}}_\beta(y) = S_F(x-y)_{\alpha\beta}.$$

7 Gauge Field Theories

Definition 7.1. A 4-vector field $A^\mu(x)$ transforms as

$$A^\mu(x) \longrightarrow \Lambda_\nu^\mu A^\nu(\Lambda^{-1} \cdot x).$$

Definition 7.2. The field strength tensor is given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Proposition 7.3. $F^{\mu\nu}$ is invariant under gauge transformations

$$A^\mu \longrightarrow A^\mu + \partial^\mu \lambda,$$

| for arbitrary $\lambda : \mathbb{R}^{3,1} \rightarrow \mathbb{R}$.

The Maxwell Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Hence by the principle of least action we can find equations of motions given by $\partial_\mu F^{\mu\nu} = 0$. The other two EoM obey the Bianchi Identities

$$\partial_\mu {}^*F^{\mu\nu} = 0,$$

where ${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

Let us try to naively take Maxwell's action and follow our recipe of trying to define variables in their conjugate momentum. Define

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \begin{cases} 0 & \text{if } \mu = 0 \\ -F^{0i} = E^i & \text{if } \mu = i \in \{1, 2, 3\} \end{cases}$$

This reflects the redundancy of our field resulting from gauge invariance. The configuration space is thus given by equivalence classes of gauge orbits. We'll take the quotient and try to perform gauge fixing, which means to pick one point from each gauge orbit. The problem with physical gauges inevitably break Lorentz invariance. This is an artefact of gauge fixing, but if the calculation is done correctly we'll recover it at the end. Otherwise we can pick covariant gauges, which preserve Lorentz invariance, such as the Lorentz gauge $\partial_\mu A^\mu = 0$.

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